ALTERNATIVE PROOF OF KEITH-ZHONG SELF-IMPROVEMENT AND CONNECTIVITY

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Abstract. We find new proofs for the celebrated theorem of Keith and Zhong that a $(1,p)$-Poincaré inequality self-improves to a $(1,p-\epsilon)$-Poincaré inequality. The paper consists of two proofs, the second of which identifies a novel characterization of Poincaré inequalities and uses it to give an entirely new proof which is closely related to Muckenhoupt weights. This new characterization, and the alternative proof, demonstrate a formal similarity between Muckenhoupt weights and Poincaré inequalities. The proofs we give are short and somewhat more direct. With them we can give the first completely transparent bounds for the quantity of self-improvement and the constants involved. We observe that the quantity of self-improvement is, for large p, directly proportional to p, and inversely proportional to a power of the doubling constant and the constant in the Poincaré inequality.

Keywords: Poincaré inequality, self-improvement, metric spaces, PI-space, analysis on metric spaces, connectivity, Muckenhoupt-weights

MSC: 30L99, 42B25, 39B72

Contents

1. Introduction

1.1. **Outline.** Our goal is two-fold. On the one hand, we wish to reprove a result by Keith and Zhong on the self-improvement of Poincaré inequalities [15], and to give explicit bounds for the quantity of self-improvement. Secondly, our goal is to draw attention to an intimate connection between the theory of Muckenhoupt weights (see [20]) and Poincaré inequalities.

We were motivatived to re-investigate the beautiful and insightful proof of the Keith-Zhong result [15] for a few reasons. Firstly, the original proof is somewhat non-intuitive. It proceeds by an abstract argument estimating distributions of certain maximal functions where the relationships between different estimates is only revealed at the very end. Further, extracting bounds from their proof seems very complicated. This was done in [12], but the bounds seemed to deteriorate for large p. The bounds we obtain below are much sharper and show that the improvement is directly proportional to p as p tends to infinity.

On the other hand, we have worked on more general applications of "self-improvement"-type methods, where much of the machinery of the original proof of Keith and Zhong becomes unnecessary [7]. Our goal is to understand whether the framework of [7] could be used to provide an even easier proof of the Keith-Zhong result. However, to achieve this goal we need new techniques, because the paper in [7] does not give sharp characterizations of Poincaré inequalities. More precisely, while those results are sharp in general, for several classes of spaces better results can be obtained, and thus we needed to develop an understanding of different characterizations.

These characterizations come in the flavor of Muckenhoupt-type conditions. Thus, an additional motivation of this paper is to study the formal similarity between Poincaré inequalities and Muckenhoupt weights. This similarity was alluded to in our prior paper [7], but we wish to make this formal analogy more precise. In the process, we obtain a new characterization of Poincaré-inequalities that clarifies the dependence of the exponent. Additionally, the relationship to Muckenhoupt-weights has been observed previously as a way of characterizing measures on $\mathbb{R}$ which admit Poincaré inequalities [3]. Thus, our results can be thought of as weaker and higher dimensional analogues of such characterizations.

This new characterization allows us to give a self-improvement result for a connectivity condition similar to the one introduced in [7], and which gives a rather different proof of Keith-Zhong self-improvement result, which is quite similar to the proof of self-improvement for Muckenhoupt weights [20]. This proof also lends to more transparent bounds for the quantity of self-improvement.

Similar results have been concurrently developed by other authors in [16]. Their methods yield more general insights into self-improvement phenomena, while this write up is restricted to classical Poincaré inequalities. Also, a careful examination of their paper seems to lead to similar bounds for the self-improvement.

We would also like to mention the recent work of Lukáš Malý on types of Lorentz-Poincaré inequalities without self-improvement, and general conditions for self-improvement for various types of Poincaré inequalities. This has yet to be published. Finally, we mention the thesis of Dejarnette [6], where he refines techniques of Keith and Zhong to prove self-improvement results for Orlicz-Poincaré inequalities.

To state the results, we will need the following terminology. For simplicity, we will consistently work with a proper metric measure spaces (X, d, μ) with a locally finite measure (i.e. $\mu(B(x, r)) < \infty$ for all open balls $B(x, r) \subset X$).

Definition 1.1. A proper metric measure space (X, d, μ) with a σ-finite Borel measure μ is said to be D-doubling if for all $0 < r$ and any $x \in X$ we have

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq D. \tag{1.2}$$

The average of a measurable function $f \colon X \to \mathbb{R}$ on a metric measure space (X, d, μ) over a set A, with $0 < \mu(A) < \infty$, is denoted by

$$f_A = \fint_A f \, d\mu = \frac{1}{\mu(A)} \int f \, dA,$$

when it makes sense, and it's local (upper) Lipschitz constant is defined as

$$\mathrm{Lip}\, f(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If $B = B(x, r)$ is a ball, we denote $CB = B(x, Cr)$ (despite the ambiguity that a ball as a set may not be uniquely defined by a center and a radius).

Definition 1.3. Let $1 \leq p < \infty$ be given. A proper metric measure space (X, d, μ) with a Radon measure μ and $\mathrm{supp}(\mu) = X$ is said to satisfy a $(1, p)$-Poincaré inequality (with constants (C, C_{PI})) if for all Lipschitz functions f and all $x \in X, 0 < r$ we have for $B = B(x, r)$

$$\fint_B |f - f_B| d\mu \leq C_{PI} r \left(\fint_{CB} \text{Lip } f^p d\mu \right)^{\frac{1}{p}}.$$

This inequality could be expressed in different generalities, but we choose this simple expression as it is sufficient. For a detailed discussion of these issues we refer to [14, 9, 10].

By an application of Hölder's inequality, we can see that for smaller p the $(1, p)$-Poincaré inequality becomes stronger. Thus, the main result of Keith-Zhong, that on a doubling complete metric measure space a $(1, p)$-Poincaré inequality implies a $(1, p - \epsilon)$-Poincaré inequality, is called a self-improvement result. For a more detailed discussion of the background and motivation we refer to [15, 7].

Theorem 1.4 (Keith-Zhong [15])**.** *Let (X, d, μ) be a proper D-doubling metric measure space with a $(1, p)$-Poincaré inequality with constants (C, C_{PI}). There exists a $\epsilon(D, p, C_{PI}) > 0$ such that for any $0 < \epsilon < \epsilon(D, p, C_{PI})$ the space admits a $(1, p - \epsilon)$-Poincaré inequality with constants $C' = C'(D, C_{PI}, \epsilon), C'_{PI} = C'_{PI}(D, p, C_{PI}, \epsilon)$.*

We can use the following bound for geodesic spaces when $C = 1$,

$$\epsilon(D, p, C_{PI}) = \frac{p}{(2^8 D^3 C_{PI})^{\frac{p}{p-1}} (4D)^{\frac{4}{p-1}} + 1},$$

for p when $2^p \geq D^3$. Bounds for smaller p are presented below in (4.32), but they are slightly more complicated.

Letting $p \to \infty$, we obtain the asymptotic estimate for the improvement $\frac{p}{2^8 C_{PI} D^3}$. This is, naturally, not a tight bound. Note that this estimate means that for larger p the improvement in Keith-Zhong becomes larger, and in fact is linearly proportional to it. We remark, that sharp bounds for the self-improvement of Muckenhoupt-weights have been studied in [13], as well as the references mentioned therein.

We present two approaches to Theorem 1.4. First, in Section 3 we modify the proof in [15] to use an induction argument, which we feel is more direct. The bounds obtained are made explicit, and are somewhat worse than the bounds we obtain at the end of the paper. Following this proof, in Section 4, we introduce a new notion of A_p-connectivity (see Definition 4.1), and p-max connectivity (see Definition 4.7).

Theorem 1.5. *For a proper metric measure space (X, d, μ) which is D-doubling the following conditions are equivalent.*

1. *X satisfies a $(1, p)$-Poincaré inequality and is D-doubling for some $D > 0$.*
2. *There is a constant X such that for every continuous f and any upper gradient g for f, then there is a constant C such that for all $x, y \in X$*

$$|f(x) - f(y)| \leq C|x - y| \left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}} \right)$$

3. *X is A_p-connected.*

These characterizations can be used to prove a Keith-Zhong-type result for connectivity, which by the previous theorem is equivalent to the self-improvement result for Poincaré inequalities.

Theorem 1.6. *If (X, d, μ) is proper A_p-connected (with constants C, C') D-doubling metric measure space, then it is $A_{p-\epsilon}$-connected for all $0 < \epsilon < \epsilon(C, C', p)$ with constants depending on ϵ, C, C', p.*

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2. Preliminary lemmas

Throughout this paper we will assume that (X, d, μ) is a proper metric measure space equipped with a Radon measure μ.

We also need a notion of a curve fragment. For more details on them see [2, 19, 7].

Definition 2.1. A curve fragment in a metric space (X, d) is a Lipschitz map $\gamma\colon K \to X$, where $K \subset \mathbb{R}$ is compact. For simplicity, we translate the set so that $\min(K) = 0$. We say the curve fragment connects points x and y if $\gamma(0) = x, \gamma(\max(K)) = y$. Further, define $\mathrm{Undef}(\gamma) = [0, \max(K)] \setminus K$. If $K = [0, \max(K)]$ is an interval we simply call γ a curve.

The length of a curve fragment is defined as

$$\mathrm{Len}(\gamma) = \sup_{x_1 \leq \cdots \leq x_n \in K} \sum_{i=1}^{n} d(\gamma(x_{i+1}), \gamma(x_i)). \tag{2.2}$$

Since γ is assumed to be Lipschitz we have $\mathrm{Len}(\gamma) \leq \mathrm{LIP}(\gamma) \max(K)$.

Analogous to curves we can define an integral over a curve fragment $\gamma\colon K \to X$. Denote

$$\sigma(t) = \sup_{x_1 \leq \cdots \leq x_n \in K \cap [0,t]} \sum_{i=1}^{n} d(\gamma(x_{i+1}), \gamma(x_i)). \tag{2.3}$$

The function $\sigma|_K$ is Lipschitz on K. Thus, it is differentiable for almost every density point $t \in K$ and for such t we set $d_\gamma(t) = \sigma'(t)$ and call it the *metric derivative* (see [1] and [2]). We define an integral of a Borel function g as follows

$$\int_\gamma g \; ds = \int_K g(\gamma(t)) \cdot d_\gamma(t) \; dt,$$

when the right-hand side makes sense. This is true, for example, if g is bounded from below or above.

By a *gap* of a curve fragment $\gamma\colon K \to X$ we mean a maximal bounded open interval (a, b) in the complement of $\mathbb{R} \setminus K$. There are at most countably many such gaps. By a *geodesic* metric space X we mean one where between each pair $x, y \in X$ there is a rectifiable curve γ connecting x to y such that $\mathrm{Len}(\gamma) = d(x, y)$.

If f is a continuous function on X, we call g an *upper gradient* for f if for every $x, y \in X$, and any rectifiable curve connecting x to y we have

$$|f(x) - f(y)| \leq \int_\gamma g \; ds.$$

We define a maximal-type operator that measures the oscillation of a locally integrable function f (see also [15]). Fix a scale parameter s and a subset $C \subset X$ and define

$$M^{\#}_{s,C} f(x) = \sup_{\substack{x \in B(y,r) \\ 0<r<s \\ y \in C}} \frac{1}{2s} \fint_{B(z,s)} |f - f_{B(z,s)}| \; d\mu$$

The following lemma is standard, but we recall the proof for the sake of completeness (see [15] and [12]).

Lemma 2.4. *Let $k \geq 5$. Assume that (X, d, μ) is a D-doubling and geodesic metric measure, and $f\colon X \to \mathbb{R}$ is a Lipschitz function such that for a ball $B = B(x, r) \subset X$ we have*

$$\frac{1}{2r}\fint_B |f - f_B|\, d\mu \leq M.$$

If $v, w \in B(x,r)$ are such that $M^{\#}_{r/k,B}f(v) \leq M$ and $M^{\#}_{r/k,B}f(w) \leq M$, then

$$|f(v) - f(w)| \leq 2^5 D^{\log_2(k)+3} M d(x,y).$$

Proof: The proof uses a standard telescoping argument, such as in [10, Chapters 4 & 5] and [9]. There are two cases $d(v,w) \leq \frac{r}{2k}$ and $d(x,y) > \frac{r}{2k}$. In the first case define $B_i = B(v, d(v,w)2^{-i})$ for $i \geq 0$, and $B_i = B(w, d(v,w)2^{1+i})$ for $i \leq -1$. Let r_i be the radius of B_i. Since the space is geodesic the intersection $B_i \cap B_{i+1}$ always contains a ball of radius $\max\{r_i/2, r_{i+1}/2\}$. We get by the assumption and doubling that

$$|f_{B_i} - f_{B_{i+1}}| \leq 2^{3-|i|} D^3 M d(x,y).$$

Next a standard telescoping argument combined with the assumption on $M^{\#}_{r/k,B}f$, doubling and continuity gives

$$|f(v) - f(w)| \leq \sum_{i\in\mathbb{Z}} 2^{3-|i|} D^3 M d(x,y) = 2^5 D^3 M d(v,w).$$

If $d(v,w) > r/(2k)$, define $B_0 = B(x,r)$, $B_i = B(v, r2^{1-i}/k)$ for $i \geq 1$, and $B_i = B(w, r2^{1+i}/k)$ for $i \leq -1$. Denote by r_i the radius of B_i. Since we are in a geodesic metric space, there is always a ball of radius $\max\{r_i/(2k), r_{i+1}/(2k)\}$ in $B_i \cap B_{i+1}$. Again, we get

$$|f_{B_i} - f_{B_{i+1}}| \leq 2^{3-|i|} D^{\log_2(k)+3} M d(v,w).$$

Thus,

$$|f(v) - f(w)| \leq \sum_{i\in\mathbb{Z}} 2^{3-|i|} D^{\log_2(k)+3} M d(v,w) = 2^5 D^{\log_2(k)+3} M d(v,w).$$

□

The following lemma follows easily from the triangle inequality.

Lemma 2.5. *Let (X,d,μ) be a D doubling metric measure space, f a locally integrable function, and $B = B(x,r)$. Then for any $a \in \mathbb{R}$*

$$\frac{1}{2r}\fint |f - f_B|\, d\mu \leq \frac{1}{r}\fint |f - a|\, d\mu.$$

We will also use a standard covering theorem from harmonic analysis (see e.g. [20, 8]).

Theorem 2.6. *(Vitali covering theorem) Let X be a metric space, $E \subset X$ a subset and $\mathcal{B}$ be a collection of balls $B(x_i, r_i)$ for $i \in I$ for some index set I (possibly uncountable) and $0 < r_i < D$ for a fixed $D > 0$ and all $i \in I$. Assume that they cover E:*

$$E \subset \bigcup_{B\in\mathcal{B}} B.$$

Then there exists a sub-collection $\mathcal{B}' \subset \mathcal{B}$ of disjoint balls, i.e. $B \cap B' = \emptyset$ for any $B, B' \in \mathcal{B}'$ if $B \neq B'$, and

$$E \subset \bigcup_{B\in\mathcal{B}'} 8B.$$

We also need some maximal function estimates. First recall, for $1 \leq p < \infty$ we define the usual class of locally integrable functions

$$L^p_{\text{loc}} = \left\{ f \text{ measurable} \,\middle|\, \int_{B(x,r)} |f|^p \, d\mu < \infty, \forall B(x,r) \subset X \right\}.$$

Definition 2.7. Let $f \in L^1_{\text{loc}}$. Define the *(centered)* maximal function at scale r as

$$M_r f(x) = \sup_{0<s<r} \fint_{B(x,s)} f \, d\mu.$$

If the subscript is dropped we obtain the Hardy-Littlewood maximal function.

$$Mf(x) = \sup_{0<s} \fint_{B(x,s)} f \, d\mu.$$

We have the following weak L^1-distributional inequality. Its proof is contained in [20].

Theorem 2.8. *(Maximal function estimate) Let (X, d, μ) be a D-measure doubling metric measure space and $s > 0$ and $B(x,r) \subset X$ arbitrary, then for any non-negative $f \in L^1$ and $\lambda > 0$ we have*

$$\mu\left(\{M_s f > \lambda\} \cap B(x,r)\right) \leq D^3 \frac{||f 1_{B(x,r+s)}||_{L^1}}{\lambda}$$

and thus for any $1 < p \leq \infty$

$$||M_s f||_{L^p(\mu)} \leq C(D,p) ||f||_{L^p(\mu)}.$$

We also need a different type of Maximal function estimate, whose proof is similar to the previous theorem, but with an additional observation. It is a "multi-scale" version of the previous inequality.

Lemma 2.9. *(Max-max estimate) Let (X, d, μ) be a D-measure doubling metric measure space and $\mathcal{C} \subset X, R > 0, \lambda \geq D^3$ be arbitrary. Let h be any locally integrable non-negative Borel-function such that $h(z) = 0$ for all $z \notin \bigcap_{c \in \mathcal{C}} B(c,R)$, and define $A = \sup_{x \in \mathcal{C}} M_R h(x)$ and the set $E_\lambda = \{z | M_R h(z) \geq \lambda A\}$. We have the following estimate for every $c \in \mathcal{C}$.*

$$M_R 1_{E_\lambda(c)} \leq \frac{D^4}{\lambda} \tag{2.10}$$

Proof: Let $c \in \mathcal{C}$ and $0 < s < R$ be arbitrary. Using Theorem 2.6, cover the set $B(c,s) \cap E_\lambda$ with a collection of balls $\mathcal{B} = \{B(x_i, 8r_i)\}$ such that $B(x_i, r_i)$ are disjoint, $x_i \in E_\lambda \cap B(c,s)$, $0 < r_i < R$ and

$$\fint_{B(x_i,r_i)} h \, d\mu \geq \lambda A.$$

We first observe that $r_i \leq s$, for otherwise

$$\begin{aligned} A \geq \fint_{B(c,\min(2r_i,R))} h \, d\mu &\geq \frac{1}{\mu(B(c,\min(2r_i,R)))} \int_{B(x_i,r_i)} h \, d\mu \\ &\geq \frac{1}{D^2} \fint_{B(x_i,r_i)} h \, d\mu \geq \frac{\lambda A}{D^2} \end{aligned}$$

This is a contradiction because $\lambda \geq D^3$. Thus $r_i \leq s$. Therefore, we get

$$\begin{aligned} \fint_{B(c,s)} 1_{E_\lambda}\, d\mu &\leq \frac{\sum_{B(x_i,8r_i)\in\mathcal{B}} \mu(B(x_i,8r_i))}{\mu(B(c,s))} \\ &\leq D^3 \frac{\sum_{B(x_i,8r_i)\in\mathcal{B}} \mu(B(x_i,r_i))}{\mu(B(c,s))} \\ &\leq \frac{D^3}{\lambda A} \frac{\sum_{B(x_i,8r_i)\in\mathcal{B}} \int_{B(x_i,r_i)} h\, d\mu}{\mu(B(c,s))} \\ &\leq \frac{D^3}{\lambda A} \frac{\int_{B(c,\min(2s,R))} h\, d\mu}{\mu(B(c,s))} \\ &\leq \frac{D^4}{\lambda A} \fint_{B(c,\min(2s,R))} h\, d\mu \leq \frac{D^4}{\lambda} \end{aligned}$$

Taking a supremum over $0 < s < R$ and noting that c is arbitrary completes the proof.

□

3. First Proof

We will rephrase the argument in [15] as a proof by induction on scales. For oscillations comparable to LIP f the result is trivial. Further if the oscillations in a given ball are comparable to LIP fM^{-n} for some $n \geq 1$ we use a dichotomy. Either the oscillations are caused by larger oscillations over a significant portion of the set, or on most scales and locations the oscillations are small. In the first case the result follows since we already have established a Poincaré inequality when oscillations are larger. In the second case we are able to cut off the high oscillations and obtain a Poincaré inequality by applying a better Lipschitz estimate obtained from Lemma 2.4. One of the main advantages of our approach is that the proof we get is significantly shorter, and also avoids a non-trivial and technical extension theorem used in [15]. This extension lemma was also avoided in the concurrent work [16].

Theorem 3.1. *(Keith-Zhong) Let (X,d,μ) be a D-doubling metric measure space with a $(1,p)$-Poincaré inequality with constants (C,C_{PI}). There exists a $\epsilon(D,p,C_{PI})$ such that for any $0 < \epsilon < \epsilon(D,p,C_{PI})$ the space (X,d,μ) also satisfies a $(1,p-\epsilon)$-Poincaré inequality with constant $C' = C'(C,p,C_{PI},\epsilon), C'_{PI} = C'_{PI}(D,p,C_{PI},\epsilon)$.*

Proof: By [4] we know that X is L-quasiconvex, and thus we can deform the metric in an L-bi-Lipschitz way to get a geodesic metric measure space. Once that is done, we can use [9] to self-improve the inequality to the form where $C = 1$. Thus, we assume for now that $C = 1$ and that the space is geodesic. This will affect the constants obtained in the end by a factor of L, and the dependence in [9]. In other words, assume the Poincaré inequality in the following form

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \leq C_{PI} r \left(\fint_{B(x,r)} \mathrm{Lip}\, f^p\, d\mu \right)^{\frac{1}{p}},$$

where $B(x,r)$ is an arbitrary ball, and f is an arbitrary Lipschitz function.

Our argument will proceed inductively down from the scale of LIP f. Consider fixed parameters $M \geq 2, C'_{PI}, N > 1$ and fix $q < p$ close enough to p, which are chosen to satisfy inequalities determined below. We will aim to prove that for any Lipschitz function f and any ball $B(x,r)$ we have

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq 2C'_{PI} r \left(\fint_{B(x,10r)} \text{Lip } f^q \, d\mu \right)^{\frac{1}{q}}.$$

Notice that we enlarge the ball on the right by a factor of ten. This factor can be removed by applying a result from [9, 14]. We will prove the result by induction on the scale of oscillation. Thus consider the following proposition, which consists of the same inequality but with an additional constraint. We need to show for every non-negative integer n the following statement.

$$\begin{aligned} \mathcal{P}_n \quad &: \quad \text{For all Lipschitz functions } f \text{ and all balls } B(x,r) \\ \frac{\text{LIP } f}{M^{n+1}} \quad &< \quad \frac{1}{2r} \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq \frac{\text{LIP } f}{M^n}. \\ &\implies \frac{\text{LIP } f}{C'_{PI} M^n} \leq \left(\fint_{B(x,10r)} \text{Lip } f^p \, d\mu \right)^{\frac{1}{p}} \end{aligned}$$

For simplicity, scale LIP $f = 1$ and denote

$$\left(\fint_{B(x,10r)} \text{Lip } f^p \, d\mu \right)^{\frac{1}{p}} = A.$$

Base case $n = 0$, i.e. $\mathcal{P}_0$: We have

$$\frac{1}{M} < \frac{1}{2r} \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu.$$

Using the $(1,p)$-Poincaré inequality we get

$$\frac{1}{MC_{PI}} \leq \left(\fint_{B(x,r)} \text{Lip } f^p \, d\mu \right)^{\frac{1}{p}}.$$

Further we get by the Lipschitz bound Lip $f \leq 1$:

$$\left(\fint_{B(x,r)} \text{Lip } f^p \, d\mu \right)^{\frac{1}{p}} \leq \left(\fint_{B(x,r)} \text{Lip } f^q \, d\mu \right)^{\frac{1}{p}}.$$

Thus

$$\frac{1}{D^{\frac{4}{q}} M^{p/q} C_{PI}^{p/q}} \leq \left(\fint_{B(x,10r)} \text{Lip } f^q \, d\mu \right)^{\frac{1}{q}}.$$

If $C'_{PI} \geq M^{p/q} C_{PI}^{p/q} D^{\frac{4}{q}}$ the desired inequality follows.

Assume $\mathcal{P}_k$ for $k < n$ and show $\mathcal{P}_n$: Take an arbitrary Lipschitz function f and a ball $B(x,r)$ such that

$$\frac{1}{M^{n+1}} < \frac{1}{2r} \fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq \frac{1}{M^n}. \tag{3.2}$$

Our main argument is that either significantly larger oscillations occur at a significant portion of the measure, or they can be cut off. First we define rigorously what we mean by the sets of large oscillations.

Fix $K = 2D^6$ and an integer N. For each $i = 1, \ldots, N$ define $\mathcal{B}_i$ to be the set of balls $B(z, s)$ with the following properties.

- $$z \in B(x, 2r)$$
- $$\frac{1}{2s} \fint_{B(z,s)} |f - f_{B(x,s)}| \, d\mu > \frac{1}{M^{n+1-i}}$$
- $$s \leq \frac{r}{5}$$

Define the sets of large oscillation as:

$$U_i = \bigcup_{B \in \mathcal{B}_i} B.$$

Note that $U_i \subset U_j$ for $i > j$. There are two cases, either one of the $\mu(U_i)$ is large or all of them are small, relative to the size of $\mu(B(x, r))$. Start with assuming that the size is above a certain threshold.

Case for some $i = 1, \ldots, N$ we have $\mu(U_i) \geq D^3 K M^{-qi} \mu(B)$: By Theorem 2.6 we can take a subcollection $\mathcal{B}_i' \subset \mathcal{B}_i$ such that $U \subset \bigcup_{B \in \mathcal{B}_i'} 80B$ and if with $B, B' \in \mathcal{B}_i'$ are distinct, then $10B \cap 10B' = \emptyset$. We get

$$\begin{aligned} \sum_{B \in \mathcal{B}_i'} \mu(10B) &\geq \sum_{B \in \mathcal{B}_i'} D^{-3} \mu(80B) \\ &\geq D^{-3} \mu(U) \geq K M^{-qi} \mu(B(x, r)). \end{aligned}$$

By the induction hypothesis for each $B \in \mathcal{B}_i'$

$$\left(\fint_{10B} \mathrm{Lip}\, f^q \, d\mu \right)^{\frac{1}{q}} \geq \frac{1}{C_{PI}' M^{n-i}}.$$

But also by disjointness

$$\sum_{B \in \mathcal{B}_i'} \int_{10B} \mathrm{Lip}\, f^q \, d\mu \leq \int_{B(x,10r)} \mathrm{Lip}\, f^q \, d\mu. \tag{3.3}$$

Finally, combining these estimates with doubling we get the following.

$$\begin{aligned} \sum_{B \in \mathcal{B}_i'} \int_{10B} \mathrm{Lip}\, f^q \, d\mu &\geq \frac{1}{{C_{PI}'}^q M^{(n-i)q}} \sum_{B \in \mathcal{B}_i'} \mu(10B) \\ &\geq \frac{1}{{C_{PI}'}^q M^{nq}} K \mu(B(x, r)) \\ &\geq \frac{1}{{C_{PI}'}^q M^{nq}} \mu(B(x, 10r)). \end{aligned} \tag{3.4}$$

The choice of $K = 2D^6$ was used on the last line. The previous estimate combined with Estimate (3.3) gives

$$\frac{1}{M^n C'_{PI}} \leq \left(\fint_{B(x,10r)} \text{Lip } f^q \; d\mu \right)^{\frac{1}{q}},$$

which completes the proof with the assumption (3.2).

Case $\mu(U_i) < D^3 K M^{-qi} \mu(B(x,r))$ for all $i = 1, \ldots, N$: First we obtain a "better" Lipschitz bound. For any $x \notin U_i \cap B(x,2r)$ and any ball $B(z,s)$ containing x with $0 < s \leq r/5$ and $z \in B(x,2r)$ we have

$$\frac{1}{2s} \fint_{B(z,s)} |f - f_{B(z,s)}| \; d\mu \leq \frac{1}{M^{n+1-i}}.$$

Thus if we define $f_i = f|_{B(x,2r) \setminus U_i}$ we have by Lemma 2.4 that f_i is $2^5 D^7 M^{-n-1+i}$-Lipschitz on $B(x,2r) \setminus U_i$. We can extend f_i to $\overline{f}_i \colon X \to \mathbb{R}$ be $2^5 D^7 M^{-n-1+i}$-Lipschitz on all of X. Further we have, almost everywhere, Lip $\overline{f}_i \leq$ Lip $f|_{B(x,2r)\setminus U_i} 1_{X \setminus U_i} + 2^5 D^7 M^{-n-1+i} 1_{U_i}$. We define an average of these functions by

$$h = \frac{1}{N} \sum_{i=1}^{N} f_i.$$

Then

$$\text{Lip } h \leq \text{Lip } f 1_{B(x,2r)\setminus U_N} + \frac{1}{N} \sum_{i=1}^{N} 2^5 D^7 M^{-n-1+i} 1_{U_i}.$$

Then, integrating over $B(x,2r)$, using the assumption $\mu(U_i) < D^3 K M^{-qi} \mu(B(x,r))$ and the property $U_i \subset U_j$ for $i > j$ we can compute the following:

$$\begin{aligned} \fint_{2B} \text{Lip } h^p \; d\mu &\leq 2^p \frac{1}{\mu(2B)} \int_{B(x,2r)\setminus U_N} \text{Lip } f^p d\mu + \sum_{i=1}^{N} \frac{2^{7p+1} D^{7p+4}}{N^p} M^{ip} M^{-qi} K M^{(-n-1)p} \\ &\leq 2^p \fint_{2B \setminus U_N} \text{Lip } f^p d\mu + \sum_{i=1}^{N} \frac{M^{N(p-q)} 2^{7p+1} D^{7p+4}}{N^p} K M^{(-n-1)p} \\ &\leq 2^{6p-5q} D^{7(p-q)} M^{-(n+1-N)(p-q)} \fint_{2B\setminus U_N} \text{Lip } f^q d\mu \\ &\quad + \frac{M^{N(p-q)} 2^{7p+1} D^{7p+4}}{N^{p-1}} K M^{-(n+1)p}. \end{aligned} \tag{3.5}$$

Note that Lip $f \leq 2^5 D^7 M^{-n-1+N}$ almost everywhere on $2B \setminus U_N$.

Making N a large fixed number and choosing q very close to p, the second term can be made arbitrarily small. This will allow us to estimate from below certain oscillations of f. To show that h has large oscillations we will prove that outside of U_1 there are still large oscillations of f. In other words we will prove

$$\frac{1}{2^5 D^4 M^{n+1}} \leq \frac{1}{4r\mu(2B)} \int_{2B\setminus U_1} |f - f_{2B\setminus U_1}| \; d\mu.$$

Assume for simplicity that $f_{2B\setminus U_1} = 0$. Using again theorem 2.6 we can cover $U \cap B(x,r)$ with balls $E_i = B(x_i, 5r_i) \subset B(x,2r)$ such that $B(x_i,r_i)$ are disjoint, $\mu(B(x_i,r_i) \cap U_1) = \frac{\mu(B(x_i,r_i))}{2}$ and $r_i \leq \frac{r}{25}$. This is possible by continuity of measure and because for any $y \in B(x,2r)$ and any $s > \frac{r}{25}$ we have

$$\mu\left(B\left(y,s\right)\right) \geq D^{-7}\mu(B(x,r)) \geq \mu(U_1)D^{-10}K^{-1}M^q.$$

Choosing M such that $M^q > 4KD^{10} = 8D^{16}$, which occurs when

$$\left(8D^{16}\right)^{\frac{1}{q}} < M. \tag{3.6}$$

Then we have

$$\mu\left(B\left(y,s\right)\right) \geq 4\mu(U_1 \cap B(y,s)),$$

and $B(y,s)$ is not in the cover. On the other hand U_1 is open, so for any $y \in U_1$ we have a s small enough with

$$\mu\left(B(y,s)\right) = \mu(U_1 \cap B(y,s)).$$

By continuity of the measure[1] there exists an s with the desired equality

$$\mu(B(x_i,r_i) \cap U_1) = \frac{\mu(B(x_i,r_i))}{2}.$$

Define $E_i^O = B(x_i,r_i) \setminus U_1$. Since $E_i \cap U_1^c \neq \emptyset$ and by the definition of U_1, we have

$$\frac{1}{10r_i}\frac{1}{\mu(E_i)}\int_{E_i}|f - f_{E_i}|d\mu \leq \frac{1}{M^n}.$$

Using doubling, the covering properties and Lemma 2.5 we get the following.

$$\begin{aligned}
\frac{1}{M^{n+1}} &\leq \frac{1}{2r}\fint_{B(x,r)}|f|\,d\mu \\
&\leq \frac{1}{2r\mu(B(x,r))}\int_{U_1\cap B(x,r)}|f|\,d\mu + \frac{1}{2r\mu(B(x,r))}\int_{B(x,r)\setminus U_1}|f|\,d\mu \\
&\leq \frac{1}{2r\mu(B(x,r))}\sum_i\int_{E_i}|f - f_{E_i^O}|\,d\mu + \frac{1}{2r\mu(B(x,r))}\sum_i\int_{E_i}|f_{E_i^O}|\,d\mu \\
&\qquad + \frac{1}{2r\mu(B(x,r))}\int_{B(x,r)\setminus U_1}|f|\,d\mu \\
&\leq \frac{2D^3}{r\mu(B(x,r))}\sum_i\int_{E_i}|f - f_{E_i}|\,d\mu + \frac{D^4}{r\mu(B(x,2r))}\int_{B(x,2r)\setminus U_1}|f|\,d\mu \\
&\leq \frac{2D^3}{r\mu(B(x,r))}\sum_i\frac{10r_i\mu(E_i)}{M^n} + \frac{2D^4}{r\mu(B(x,2r))}\int_{B(x,2r)\setminus U_1}|f|\,d\mu \\
&\leq \frac{20D^6|U|}{M^n\mu(B)} + \frac{2D^4}{r\mu(2B)}\int_{2B\setminus U_1}|f|\,d\mu \\
&\leq \frac{20D^9KM^{-q}}{M^n} + \frac{2D^4}{r}\fint_{2B\setminus U_1}|f|\,d\mu
\end{aligned}$$

Assume next $40D^9KM^{-q-n} < M^{-n-1}$. This is possible if $q > 1$ and (by the choice of K)

$$\left(80D^{15}\right)^{\frac{1}{q-1}} < M.$$

[1]By an argument in [5] in s, the function $g_x\colon\, s \to \mu(B(x,s))$ is continuous with an efficient bound on the modulus of continuity. For a similar argument see also the book [12].

Thus we can absorb the first term on the bottom row to the left-hand side and obtain the following

$$\frac{1}{2M^{n+1}} \leq \frac{2D^4}{r} \fint_{2B\setminus U_1} |f| d\mu, \tag{3.7}$$

which gives the desired estimate

$$\frac{1}{2^5 D^4 M^{n+1}} \leq \frac{1}{8r\mu(2B\setminus U_1)} \int_{2B\setminus U_1} |f - f_{2B\setminus U_1}| \, d\mu.$$

By assuming (3.6), we have $KD^3M^{-q} \leq \frac{1}{2}$. Thus, we get $\mu(U_1) \leq \frac{1}{2}\mu(B(x,r)) \leq \frac{1}{2}\mu(B(x,2r))$ and thus for $h = f_{2B\setminus U_1}$

$$\frac{1}{8r\mu(2B\setminus U_1)} \int_{2B\setminus U_1} |f - f_{2B\setminus U_1}| \, d\mu \leq \frac{1}{r} \fint |h - h_{2B}| \, d\mu. \tag{3.8}$$

In particular, combining (3.7) and (3.8) we get

$$\frac{1}{4r} \fint |h - h_{2B}| \, d\mu \geq \frac{1}{2^7 D^4 M^{n+1}}$$

By the $(1,p)$-PI inequality

$$\frac{1}{2^{7p} C_{PI}^p D^{4p} M^{p(n+1)}} \leq \fint_{2B} \text{Lip } h^p d\mu.$$

Combining this with the estimate for Lip h in (3.5) we see that

$$\begin{aligned}\frac{1}{2^{7p} C_{PI}^p D^{4p} M^{p(n+1)}} &\leq 2^{6p-5q} D^{7(p-q)} M^{-(n+1-N)(p-q)} \fint_{2B\setminus U_N} \text{Lip } f^q d\mu \\ &+ \frac{M^{N(p-q)} 2^{7p+1} D^{7p+4}}{N^{p-1}} K M^{-(n+1)p}.\end{aligned}$$

If

$$\frac{M^{N(p-q)} 2^{7p+1} D^{7p+4}}{N^{p-1}} K \leq \frac{1}{2^{7p+1} C_{PI}^p D^{4p}},$$

we can absorb the second term to the left-hand side. This can be attained by first fixing N large, and then taking q very close to p. See the discussion below for exact bounds. Using this we get

$$\begin{aligned}\frac{1}{2^{7p+1} C_{PI}^p D^{4p} M^{p(n+1)}} &\leq 2^{6p-5q} D^{7(p-q)} M^{-(n+1-N)(p-q)} \fint_{2B\setminus U_N} \text{Lip } f^q d\mu \\ &\leq 2^{6p-5q+1} D^{7(p-q)+3} M^{-(n+1)(p-q)} M^{N(p-q)} \fint_{10B} \text{Lip } f^q \, d\mu.\end{aligned}$$

Moving terms around, we get

$$\frac{1}{2^{13p-5q+2} C_{PI}^p D^{4p+7(p-q)+3} M^{N(p-q)} M^{q(n+1)}} \leq \fint_{10B} \text{Lip } f^q \, d\mu.$$

If

$$C_{PI}'^q \geq 2^{14p-5q+1} C_{PI}^p D^{11p-7q+3} M^{N(p-q)},$$

the result follows once we check that its possible to choose M, q, N, C'_{PI} so that the required six inequalities hold. This is done in the final step.

Choosing parameters:
We have six inequalities that need to be satisfied for the aforementioned proof.

(1) $C'_{PI} \geq M^{\frac{p}{q}} C_{PI}^{p/q} D^{4/q}$
(2) $M \geq 2$
(3) $M > \left(2^3 D^{16}\right)^{\frac{1}{q}}$
(4) $M > \left(80 D^{15}\right)^{\frac{1}{q-1}}$
(5) $\frac{1}{2^{7p+1} C_{PI}^p D^{4p}} \geq \frac{M^{N(p-q)} 2^{7p+2} D^{7p+10}}{N^{p-1}}$
(6) $C'^q_{PI} \geq 2^{14p-5q+1} C_{PI}^p D^{11p-7q+3} M^{N(p-q)}$

Since we do not expect the Poincaré to improve much, and we only want rough bounds, we will assume $q-1 > (p-1)/2$, and set

$$M = \max\left(2, \left(2^8 D^{16}\right)^{\frac{2}{p-1}}\right) \tag{3.9}$$

to guarantee the lower bounds for M.

Next, we choose N and q in order to get the fourth estimate. Taking a power $1/p$ on both sides and re-arranging terms gives

$$\frac{1}{2^{14+3/p} M C_{PI} D^{11+14/p}} \geq \frac{M^{(1-q/p)N}}{N^{1-1/p}}$$

This is satisfied if $N^{1-1/p} \geq 2^{15+3/p} M C_{PI} D^{11+14/p} 2$ and $M^{(1-q/p)N} \leq 2$. First, choose

$$N = \left(2^{15+3/p} M C_{PI} D^{11+14/p}\right)^{\frac{p}{p-1}}. \tag{3.10}$$

Then, choose q close enough to p so that $M^{(1-q/p)N} \leq 2$. This is possible if

$$p - q \leq \frac{p}{\log_2(M) N}. \tag{3.11}$$

Finally, we can choose C'_{PI} to satisfy the first and fifth inequality.

$$C'_{PI} = M^{p/q} C_{PI}^{p/q} D^{25} 2^{30} \leq 2^{31+14/(p-1)} D^{32/(p-1)} C_{PI}^{p/q} \tag{3.12}$$

□

Finally, to understand the asymptotics as $p \to \infty$, we can choose $M = 2$, once p is sufficiently large. Then, for $p \geq 4$ we can set $N = \left(2^{16} C_{PI} D^{12}\right)^{\frac{p}{p-1}}$. Finally, this gives the bound

$$p - q \leq \epsilon(D, p, C_{PI}) = \frac{p}{(2^{16} C_{PI} D^{12})^{\frac{p}{p-1}}}.$$

That is, asymptotically, the improvement is linearly proportional to p, with ratio $\frac{1}{2^{16} C_{PI} D^{12}}$. In other words, for larger p the improvement is greater.

If p is not sufficiently large, then the bounds become more complex, and we can set

$$M = 2^{1+14/(p-1)} D^{32/(p-1)}$$
$$N = \left(2^{18+15/(p-1)} D^{15+42/(p-1)} C_{PI}\right)^{\frac{p}{p-1}}$$

and

$$\epsilon(D,p,C_{PI}) = \frac{p}{\left(1+\frac{15}{p-1}+\log_2(D)\frac{32}{p-1}\right)\left(2^{18+14/(p-1)} D^{15+42/(p-1)} C_{PI}\right)^{\frac{p}{p-1}}}.$$

4. Relation to connectivity

We will use the following definition of A_p-connectivity.

Definition 4.1. Let $C, C' > 0, p \geq 1$. We say that a metric measure space (X,d,μ) is A_p-connected (with constants (C,C')) if for every $\tau > 0$ and every $x, y \in X$ with $d(x,y) = r > 0$, and every Borel-measurable and non-negative g, there exists a $L > 0$ and a 1-Lipschitz curve $\gamma\colon [0,L] \to X$ such that

(1) $\gamma(0) = x$
(2) $\gamma(L) = y$
(3) $\mathrm{Len}(\gamma) \leq Cr$
(4)

$$\int_\gamma g \leq C' r \left(M_{Cr}(g^p)(x)^{\frac{1}{p}} + M_{Cr}(g^p)(y)^{\frac{1}{p}}\right) \tag{4.2}$$

We choose the term A_p-connected to draw an analogy to the definition of A_p-weights. We say $\mu \in A_p(\lambda)$, where λ is Lebesgue measure on $\mathbb{R}^n$ if one of the following equivalent conditions holds.

(1) Maximal function bound. There is a constant $C > 0$ such that for every $f \in L^p$ we have

$$\left(\int (Mf)^p \, d\mu\right)^{\frac{1}{p}} \leq C \left(\int f^p \, d\mu\right)^{\frac{1}{p}} \tag{4.3}$$

(2) Integral bound. $\mu = \omega\lambda$, where ω, ω^{1-p} are locally integrable and there is a $C > 0$ such that for every ball $B(x,r)$

$$\left(⨍_B w \, d\lambda\right)\left(⨍_B w^{1-p} \, d\lambda\right)^{\frac{1}{p-1}} \leq C \tag{4.4}$$

(3) Average bound. For some $C > 0$ and for any f locally integrable and any ball $B(x,r)$

$$⨍_B f \, d\lambda \leq C \left(\frac{1}{\mu(B)} \int_B f^p w \, d\lambda\right)^{\frac{1}{p}}. \tag{4.5}$$

Further, all of these imply that a quantitative absolute continuity holds. By this, we mean that there is a constant $C > 0$ such that for all $B(x,r)$ and all $E \subset B(x,r)$ we have

$$\frac{\lambda(E)}{\lambda(B(x,r))} \leq C \left(\frac{\mu(E)}{\mu(B(x,r))}\right)^{\frac{1}{p}}. \tag{4.6}$$

It is subtle, that this quantitative absolute continuity is *not* equivalent to being an A_p-weight. In fact, by work in [17, 18] the condition (4.6) characterizes so called $A_{p,1}$-weights. While the A_p-conditions characterize boundedness of the Hardy-Littlewood maximal function M from L^p to L^p, the $A_{p,1}$-condition characterizes boundedness from $L^p \to L^{p,\infty}$. It is known, that $A_p \subset A_{p,1}$ strictly. Further, the $A_{p,1}$-condition does not improve to $A_{q,1}$ for any $q < p$.

Our definition of A_p-connected is analogous to the average bound (4.5). Namely, replace λ by $\mathcal{H}^1|_\gamma$ and the right-hand side by a maximal function bound. The measure $\mathcal{H}^1_\gamma$ is the 1-dimensional Hausdorff measure on the image of γ. The formal difference is that the A_p-connectivity additionally presumed the *existence* of some curve γ such that the estimate holds. In a sense, the A_p-connectivity is an A_p-weight with respect to one-dimensional Hausdorff measure on some curve.

The analogue of (4.4) is a $(1,p)$-Poincaré inequality, and the analogues of (4.6) is the following notion of p-max connectivity. There does not seem to be a natural analogy for the estimate (4.4), and we leave it open if there is any estimate related to Poincaré inequalities which plays the same role.

Definition 4.7. Let $C, C' > 0, p \geq 1$. We say that a metric measure space (X,d,μ) is p-max-connected (with constants (C,C')) if for every $\tau > 0$ and every $x, y \in X$ with $d(x,y) = r$, and every Borel-measurable E such that $M_{Cr}(1_E)(x) < \tau$ and $M_{Cr}(1_E)(y) < \tau$, there exists a $L > 0$ and a 1-Lipshitz curve $\gamma\colon [0,L] \to X$ such that[2]

(1) $\gamma(0) = x$
(2) $\gamma(L) = y$
(3) $\text{Len}(\gamma) \leq Cr$
(4)

$$\int_\gamma 1_E \, ds \leq C' \tau^{\frac{1}{p}} r. \tag{4.8}$$

Theorem 4.9. *For a proper metric measure space (X,d,μ) the following conditions are equivalent.*

(1) *X satisfies a $(1,p)$-Poincaré inequality and is D-doubling for some $D > 0$.*
(2) *If g is a weak upper gradient for a continuous f, then there is a constant C such that for all $x, y \in X$*

$$|f(x) - f(y)| \leq C|x-y| \left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}} \right)$$

(3) *X is A_p-connected.*

Proof: That the first and second are equivalent follows from a classical result, which is presented for example in [11, Lemma 5.15]. On the other hand the third condition implies the second, since it implies the existence of a curve connecting x to y where the integral is controlled by the desired quantity. It remains to show that the first two conditions imply the last one.

Assume thus that (X,d,μ) satisfies a $(1,p)$-Poincaré inequality and the second condition, and let g be an arbitrary non-negative Borel-function such that g^p is locally integrable and fix $x, y \in X$. To fix constants, assume the Poincaré inequality in the form

$$\fint_B |f - f_B| d\mu \leq C_{PI} r \left(\fint_B \text{Lip } f^p d\mu \right)^{\frac{1}{p}}$$

and the second condition as

$$|f(x) - f(y)| \leq C|x-y| \left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}} \right)$$

.

We will construct γ such that

$$\text{Len}(\gamma) \leq 5Cd(x,y)$$

and

[2] If we chose to use curve fragments, we could instead quantify the sizes of gaps and assume that γ avoids the set E. Such variants of these definitions are discussed in [7].

$$\int_\gamma g\, ds \leq 4Cd(x,y)\left(M_{Cr}(g^p)(x)^{\frac{1}{p}} + M_{Cr}(g^p)(y)^{\frac{1}{p}}\right).$$

First, take a lower semi-continuous $\overline{g} \geq g$ such that

$$M_{Cr}(\overline{g}^p)(x)^{\frac{1}{p}} + M_{Cr}(\overline{g}^p)(y)^{\frac{1}{p}} \leq 2M_{Cr}(g^p)(x)^{\frac{1}{p}} + 2M_{Cr}(g^p)(y)^{\frac{1}{p}} \tag{4.10}$$

This is possible by a standard approximation argument applied to g restricted on different annuli. Next, define for every $N > 0$ and $\epsilon > 0$ a function $\overline{g_{N,\epsilon}} = \min(\overline{g} + \epsilon, N)$. Then

$$M_{Cr}(\overline{g_{N,\epsilon}}^p)(x)^{\frac{1}{p}} + M_{Cr}(\overline{g_{N,\epsilon}}^p)(y)^{\frac{1}{p}} \leq 2M_{Cr}(g^p)(x)^{\frac{1}{p}} + 2M_{Cr}(g^p)(y)^{\frac{1}{p}} + 2\epsilon. \tag{4.11}$$

Now, define for $z \in X$ the set $\Gamma_{x,z}$ as the set of all rectifiable curves starting at x and ending at z. Further, define a function by

$$\mathcal{F}_{N,\epsilon}(z) = \inf_{\gamma \in \Gamma_{x,z}} \int_\gamma \overline{g_{N,\epsilon}}\, ds. \tag{4.12}$$

This function is bounded and continuous, since PI-spaces are L-quasiconvex for some $L = L(C_{PI}, D)$. It is also easy to see that $\overline{g_{N,\epsilon}}$ is an upper gradient for $\mathcal{F}_{N,\epsilon}$. Next, by the second condition we have

$$|\mathcal{F}_{N,\epsilon}(y) - \mathcal{F}_{N,\epsilon}(x)| \leq 2C|x-y|\left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right) + 2C|x-y|\epsilon. \tag{4.13}$$

Thus, there is a curve $\gamma_{N,\epsilon}$ such that $\gamma_{N,\epsilon}$ connects x to y and

$$\int_{\gamma_{N,\epsilon}} \overline{g_{N,\epsilon}}\, ds \leq 2C|x-y|\left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right) + 3C|x-y|\epsilon.$$

Assume now $\epsilon > M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}$ is arbitrary. Then since $\overline{g_{N,\epsilon}} \geq \epsilon$, we get

$$\begin{aligned}
\epsilon \mathrm{Len}(\gamma_{N,\epsilon}) &\leq \int_{\gamma_{N,\epsilon}} \overline{g_{N,\epsilon}}\, ds \\
&\leq 2C|x-y|\left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right) + 3C|x-y|\epsilon \qquad (4.14)\\
&\leq 5C|x-y|\epsilon. \qquad (4.15)
\end{aligned}$$

Thus $\mathrm{Len}(\gamma_{N,\epsilon}) \leq 5C|x-y|$. Next, let $N \to \infty$, and choose a subsequential limit γ_ϵ of the curves $\gamma_{N,\epsilon}$. Then for every N, using lower semi-continuity, we get

$$\begin{aligned}
\int_{\gamma_\epsilon} \overline{g_{N,\epsilon}}\, ds &\leq \liminf_{M\to\infty} \int_{\gamma_M} \overline{g_{N,\epsilon}}\, ds \qquad (4.16)\\
&\leq \liminf_{M\to\infty} \int_{\gamma_M} \overline{g_{M,\epsilon}} \qquad (4.17)\\
&\leq 2C|x-y|\left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right) + 3C|x-y|\epsilon \qquad (4.18)
\end{aligned}$$

Now, letting $N \to \infty$, we get

$$\int_{\gamma_\epsilon} \overline{g} + \epsilon\, ds \leq 2C|x-y|\left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right) + 3C|x-y|\epsilon. \tag{4.19}$$

Letting $\epsilon \to \left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right)$, taking a sub-sequential limit γ of γ_ϵ and using lower semi-continuity of $\overline{g}$ we get the desired estimate

$$\int_\gamma g\ ds \leq \int_\gamma \overline{g} \leq 5C|x-y| \left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right).$$

We remark, that this final limiting process is only necessary if $\left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right) = 0$. Otherwise, we could just set $\epsilon = \left(M_{Cr}(g(x)^p)^{\frac{1}{p}} + M_{Cr}(g(y)^p)^{\frac{1}{p}}\right)$.

□

Next, we present a different argument for Keith-Zhong self-improvement based on ideas from [18], where the authors show general self-improvement phenomenon for Maximal-function estimates. There, a crucial role is played by a sub-multiplicative quantity. For us the relevant quantity is the following.

Let $x, y \in X$ be given and denote by $r = d(x,y)$. Denote by

$$\mathcal{E}_{x,y,\tau,C} = \{E \subset B(x,Cr) \cup B(y,Cr) | E \text{ open and } M_{Cr}1_E(x) < \tau, M_{Cr}1_E(y) < \tau\} \tag{4.20}$$

the set of admissible obstacles. Denote by $\Gamma_{x,y}^C$ the set of rectifiable curves γ parametrized by length on the interval $[0, \mathrm{Len}(\gamma)]$ such that $\gamma(0) = x, \gamma(\mathrm{Len}(\gamma)) = y$. Then define

$$\alpha(\tau, C) = \sup_{x,y \in X} \sup_{E \in \mathcal{E}_{x,y,\tau,C}} \inf_{\gamma \in \Gamma_{x,y}^C} \frac{1}{d(x,y)} \int_\gamma 1_E\ ds. \tag{4.21}$$

Note, for example, that $\alpha(1, C) = 1$ for a geodesic space.

Also, define the quantity

$$\beta(C) = \limsup_{\tau \to 0} \frac{\log(\alpha(\tau, C))}{\log(\tau)}. \tag{4.22}$$

In fact, we could replace the limit superior by a normal limit due to sub-multiplicativity. It is easy to see, that for any C and any $p > 1/\beta(C)$ we have that the space is p-max connected. Further, we have the following result.

Theorem 4.23. *If (X, d, μ) is a proper D-doubling and p-max-connected metric measure space with constants (C, C'), then it satisfies a $(1, q)$-Poincaré inequality for every $q > p$.*

Proof: We show that the space is A_q-connected for every $q > p$ with constants (C_∞, D_∞) (see below for definitions of these), from which the result follows by Theorem 1.5. The core steps of the argument are the same as [7, Theorem 3.10], and we refer for more details to that proof. Here, we state only the main outline. Without loss of generality, we only need to verify A_q-connectivity for continuous functions.

First, by a limiting argument it is enough to prove the following inductive statement. Choose $M = \max\left((4^{p+q+2}C'^p D^4)^{\frac{1}{p-q}}, 4D^{\frac{4}{p}}\right)$ and $\delta = C'\left(\frac{4^{q+2}D^4}{M^q}\right)^{\frac{1}{p}} \leq \frac{1}{2M}$. Abbreviate $D_n = (CM + CM(M\delta) + \cdots + CM(M\delta)^{n-1})$ and $C_n = (C + C\delta + \cdots + C(\delta^{n-1}))$, and $D_\infty = \lim_{n\to\infty} D_n$ and $C_\infty = \lim_{n\to\infty} C_n$. We show the following induction statement, from which the A_q connectivity can be deduced.

$\mathcal{P}_n$: *For every $v, w \in X$ with $d(v,w) = r$, if h is a continuous locally bounded function supported on $B(v, C_\infty r) \cap B(w, C_\infty r)$ with*

$$(M_{C_\infty r}h^q)(v)^{\frac{1}{q}} + (M_{C_\infty r}h^q)(w)^{\frac{1}{q}} < 1, \tag{4.24}$$

then there is a 1-Lipschitz curve fragment $\gamma\colon K \to X$ *with*

$$\mathrm{Len}(\gamma) \leq C_n r, \tag{4.25}$$

$$\int_\gamma h\ ds \leq D_n r, \tag{4.26}$$

$$|\mathrm{Undef}(\gamma)| \leq \delta^n d(v,w), \tag{4.27}$$

and for all gaps (a_i, b_i) *of* γ *we have*

$$(M_{C_\infty d(\gamma(a_i),\gamma(b_i))} h^q(\gamma(a_i)))^{\frac{1}{q}} + (M_{C_\infty d(\gamma(a_i),\gamma(b_i))} h^q(\gamma(b_i)))^{\frac{1}{q}} < M^n. \tag{4.28}$$

In the following, we assume v, w, r, E, h to be as in the statement of $\mathcal{P}_n$.

(1) **Base Case $\mathcal{P}_1$:** Let $E_M = \{M_{C_\infty r} h^q \geq M^q/4^q\}$. By Lemma 2.9 applied to $\mathcal{C} = \{v, w\}$ and $R = C_\infty r$, we have $M_{C_\infty r} 1_{E_M}(v) < 2{\cdot}4^q D^4/M^q$ and $M_{C_\infty r} 1_{E_M}(w) < 2{\cdot}4^q D^4/M^q$. Thus, by p-max-connectivity, there is a 1-Lipschitz curve $\gamma\colon [0, L] \to X$ connecting v to w such that $\mathrm{Len}(\gamma) \leq C_1 r$ and

$$\int_\gamma 1_E\ ds \leq C' \left(\frac{4^{q+1} D^4}{M^q}\right)^{\frac{1}{p}} r.$$

Parametrizing by unit speed and restricting this curve to the complement of $\gamma^{-1}(E_M)$ we obtain the desired curve fragment. Since the curve fragment avoids E_M, we have the desired maximal function estimate (4.28). The integral bound (4.26) and length estimate (4.25) follow since h is continuous and $h \leq M$ on the complement of E_M.

(2) **Assume $\mathcal{P}_n$ and show $\mathcal{P}_{n+1}$**: First, by $\mathcal{P}_n$, we can find a curve fragment $\gamma'\colon K' \to X$ such that it satisfies estimates (4.25), (4.26),(4.27) and (4.28) with n. We will next "fill in" the gaps of γ', and slightly dilate them, in order to construct our desired curve fragment γ.

First, denote by (a_i', b_i') the disjoint gaps in γ'. Dilate each of them by a factor $Cd(\gamma'(a_i), \gamma(b_i'))/|b_i - a_i|$ in order to obtain a 1-Lipschitz curve fragment, denoted by the same symbol in order to reduce clutter, $\gamma'\colon K' \to X$. This curve fragment will have the same length, will be 1-Lipschitz and $\min(K') = 0, \max(K') \leq C_n r + C\delta^n r \leq C_{n+1} r$. Further, the curve fragment satisfies the same estimates (4.28) and (4.26). Denote $|b_i' - a_i'| = d_i$, and thus with the dilation factored in we have $d(\gamma'(a_i), \gamma(b_i')) = d_i/C$ and

$$\sum_i d_i \leq C\delta^n r. \tag{4.29}$$

For details on such a dilation procedure see [7].

Now, for every gap in this new curve fragment we have

$$(M_{C_\infty d_i/C} h^q(\gamma'(a_i')))^{\frac{1}{q}} + (M_{C_\infty d_i/C} h^q(\gamma'(b_i')))^{\frac{1}{q}} < M^n.$$

Thus, by a scaled version of $\mathcal{P}_1$, we can find a curve fragment $\gamma_i\colon K_i \to X$ connecting $\gamma'(a_i)$ to $\gamma'(b_i)$ with $\min(K_i) = 0$, $\mathrm{Len}(\gamma_i) \leq C_1 d_i$, $\int_{\gamma_i} h\ ds \leq M^n D_1 d_i/C \leq M^{n+1} d_i$ and for every gap (a_j^i, b_j^i) of γ_i' we have

$$(M_{C_\infty d(\gamma_i(a_j^i),\gamma_i(b_j^i))} h^p(\gamma_i(a_j^i)))^{\frac{1}{p}} + (M_{C_\infty d(\gamma_i(a_j^i),\gamma_i(b_j^i))} h^p(\gamma_i(b_j^i)))^{\frac{1}{p}} < M^{n+1}. \tag{4.30}$$

Extending by a constant path at the end, we can assume that $\max(K_i) = C_1 d_i$.

We form $K = K' \cup \bigcup_i a_i' + K_i$ and defining $\gamma(t) = \gamma'(t)$ if $t \in K'$, and $\gamma(t) = \gamma_i(t - a_i)$ for $t \in a_i + K_i$. Clearly γ is 1-Lipschitz, and thus $\mathrm{Len}(\gamma) \leq \max(K) = \max(K') \leq C_{n+1} r$ (establishing

(4.25)). Next, $\text{Undef}(\gamma) = \bigcup_i a_i' + \text{Undef}(\gamma_i)$, and thus $|\text{Undef}(\gamma)| \leq \sum_i \delta d_i/C \leq \delta^n r$, which gives (4.27). Finally, equation (4.30) and the previous observation on the gaps gives (4.28). The final estimate (4.26) follows from (4.29) and

$$\int_\gamma h\ ds = \int_{\gamma'} h\ ds + \sum_i \int_{\gamma_i} h\ ds \leq D_n r + \sum_i M^{n+1} d_i \leq D_{n+1} r.$$

This concludes the induction step and thus the proof.

□

Theorem 4.31. *If (X, d, μ) is D-doubling and A_p-connected (with constants C, C'), then it is $A_{p-\epsilon}$-connected for all $0 < \epsilon < \epsilon(D, C, C', p)$ with constants depending on ϵ, C, C', p.*

Remark: Note that p-max-connectivity may fail to self-improve, as can be seen by considering the space arising from gluing two copies of $\mathbb{R}^n$ along the origin.

Proof: We can assume by a bi-Lipschitz deformation that the space is geodesic. This will change the constants involved in the theorem, and thus the dependence of the self-improvement. Recall the definitions of α, β in (4.21) and (4.22). As discussed above, it is sufficient to show that there is an $\epsilon(C, C', p)$, such that the space has $\beta(C'') > 1/p + \eta(C, C', p)$ for some C'', from which the result follows by Theorem 4.23.

Clearly the space is p-max-connected. Thus, we would expect something like $\alpha(M\tau, L) \leq C'' M^{1/p} \alpha(\tau, L)$ for some constant L, C''. We prove a slightly better estimate, that for some $k \in \mathbb{N}, M \geq 2$, $0 < \delta < 1$ and for all $L \geq 1$ and $\tau < M^{-k}$ we have

$$\alpha(\tau, C + L\delta) \leq \delta \max_{i=1,\ldots,k} M^{-i/p} \alpha(M^i \tau, L).$$

Define $C_n = \sum_{i=0}^{n-1} C\delta^i + \delta^n$, and $C_\infty = \lim_{n\to\infty} C_n$. Iterating this estimate starting with $L = 1$ gives for $N \geq 2$

$$\alpha(M^{-kN}, C_{N-1}) \leq \delta^{N-1} M^{-k(N-1)/p} \alpha(1, 1) = \delta^{N-1} M^{-k(N-1)/p}.$$

Thus, if $M^{-k(N+1)} < \tau \leq M^{-kN}$, then $\log_M(\tau)/(-k) - 2 \leq N - 1 < \log_M(\tau)/(-k) - 1$, and

$$\begin{aligned} \alpha(\tau, C_\infty) \leq \alpha(M^{-kN}, C_{N-2}) &\leq \delta^{N-1} M^{-k(N-1)/p} \alpha(1, 1) \\ &\leq \delta^{\log_M(\tau)/(-k)-2} M^{-k(\log_M(\tau)/(-k)-2)/p} \\ &\leq \delta^{-2} M^{2k/p} \tau^{1/p + \log_M(\delta)/(-k)}. \end{aligned}$$

This gives the estimate $\beta > \frac{1}{p} + \frac{-\log_M(\delta)}{k}$, which gives a Poincaré inequality for $q > \frac{kp}{k - \log_M(\delta)p}$, which proves the Keith-Zhong result.

Next, fix $M = \max\{2^p, D^3\}, 0 < \delta < 1, k \geq 1$ to be determined later. Let $x, y \in X$ and $E \in \mathcal{E}_{x,y,C+\delta L,\tau}$ be arbitrary. Assume $\tau < M^{-k}$. Define $E_i = \{z | M_{\delta Lr} 1_E > M^i \tau\}$ for $i = 1, \ldots, k$.

$$h = \frac{1}{k} \sum_{i=1}^k M^{i/p} 1_{E_i}.$$

Note that $E_i \subset E_j$ for $i < j$. Now, take an aritrary $0 < s < Cr$ and for $z = x, y$, and apply Lemma 2.9. In order to apply it, we need $M \geq D^3$.

$$\begin{aligned}
\fint_{B(z,s)} h^p \, d\mu &< \frac{2}{k^p} \sum_{i=1}^k M^i \frac{\mu(E_i \cap B(z,s))}{\mu(B(z,s))} \\
&\leq \frac{2}{k^p} \sum_{i=1}^k M^i M 1_{E_i}(z) \\
&\leq \frac{2D^4}{k^{p-1}} \left(M1_E(x) + M1_E(y)\right) \\
&\leq \frac{4D^4}{k^{p-1}}
\end{aligned}$$

Thus $(M_{Cr}h^p(x))^{\frac{1}{p}} + (M_{Cr}h^p(y))^{\frac{1}{p}} < \frac{2(4D^4)^{\frac{1}{p}}}{k^{\frac{p-1}{p}}}$. And by A_p-connectivity, there is a curve γ such that

$$\mathrm{Len}(\gamma) \leq Cr,$$

and

$$\int_\gamma h \, ds < \frac{2C'(4D^4)^{\frac{1}{p}}}{k^{\frac{p-1}{p}}} r.$$

Thus, there must be some index i such that

$$\int_\gamma 1_{E_i} M^{i/p} \, ds < \frac{2C'(4D^4)^{\frac{1}{p}}}{k^{\frac{p-1}{p}}} r.$$

Now, abbreviate $\delta = \frac{2C'(4D^4)^{\frac{1}{p}}}{k^{\frac{p-1}{p}}}$.

First parametrize γ by unit speed on the interval $[0, \mathrm{Len}(\gamma)]$ to be a 1-Lipschitz curve. Then, restrict the domain of γ to a large compact set K such that $\min(K) = 0, \max(K) = \mathrm{Len}(\gamma)$,

$$\gamma' = \gamma|_K,$$

and $\gamma'(K) = \gamma(K) \cap E_i = \emptyset$. Further, choose K large so that $|[0, \mathrm{Len}(K)] \setminus K| = |\mathrm{Undef}(\gamma')| \leq \delta r$.

Let (a_j', b_j') be the gaps of γ', and denote by $d_j' = d(\gamma(b_j'), \gamma(a_j'))$. By assumption, we have

$$\sum_j d_j' \leq \sum_j |b_j' - a_j'| \leq \delta M^{-i/p} r.$$

Now, re-scale each gap similar to Theorem 4.23, to obtain a curve fragment $\gamma'' \colon K'' \to X$, which is 1-Lipschitz and such that $\min(K'') = 0$ and $\gamma''(\min(K'')) = x, \gamma''(\max(K'')) = y$. Further, we can require that for all the gaps (a_j'', b_j'') we have the estimate $Ld(\gamma(a_j''), \gamma(b_j'')) = |b_j'' - a_j''|$. Index the gaps so that $d_j' = d(\gamma(a_j''), \gamma(b_j''))$. We obtain the estimate

$$\max(K'') \leq \mathrm{Len}(\gamma') + L\delta r \leq (C + L\delta)r.$$

For each gap, since $\gamma''(K'') \cap E_i = \emptyset$, we have

$$M_{Ld_j} 1_E(\gamma(a_j'')) < M^i \tau \text{ and } M_{Ld_j} 1_E(\gamma(b_j'')) < M^i \tau.$$

Thus, by the definition of $\alpha(M^i\tau, L)$, there exists 1-Lipschitz curves $\gamma_j'' \colon [0, Ld_j'] \to X$ connecting $\gamma''(a_j'')$ to $\gamma''(b_j'')$ of length at most $Ld_j' = |b_j'' - a_j''|$ with

$$\int_{\gamma_j''} 1_E \, ds \leq d_j' \alpha(M^i \tau, L).$$

Finally, define $\gamma''' \colon [0, \max(K'')] \to X$ by $\gamma'''(t) = \gamma''(t)$, when $t \in K''$, and $\gamma'''(t) = \gamma_j''(t - a_j'')$, when $t \in (a_j'', b_j'')$.

Clearly γ''' is 1-Lipschitz. Thus we have,

$$\mathrm{Len}(\gamma''') \leq \max(K'') \leq (C + L\delta)r.$$

Further,

$$\begin{aligned}
\int_{\gamma'''} 1_E \, ds &= \int_{\gamma'} 1_E \, ds + \sum_j \int_{\gamma_j''} 1_E \, ds \\
&= \sum_j \int_{\gamma_j''} 1_E \, ds \\
&\leq \sum_j d_j' \alpha(M^i \tau, L) \\
&\leq \delta M^{-i/p} \alpha(M^i \tau, L) r \\
&\leq \delta r \max_{i=1,\ldots,k} M^{-i/p} \alpha(M^i \tau, L)
\end{aligned}$$

Thus, taking suprema over all sets $E \in \mathcal{E}_{x,y,C+\delta L,\tau}$ and all pairs x, y gives

$$\alpha(\tau, C + \delta L) \leq \delta \max_{i=1,\ldots,k} M^{-i/p} \alpha(M^i \tau, L)$$

We required that $\delta = \frac{2C'(4D^4)^{\frac{1}{p}}}{k^{\frac{p-1}{p}}} < 1$, which is obtained once $k = (4C')^{\frac{p}{p-1}}(4D)^{\frac{4}{p-1}}$.

□

Finally, with the above choice of k, we see what the bound for q is. For any p we have

$$q > p - \frac{\log_{\max(2^p, D^3)}(2)p}{(4C')^{\frac{p}{p-1}}(4D)^{\frac{4}{p-1}} + \log_{\max(2^p, D^3)}(2)p} p,$$

and thus a bound for ϵ of Theorem 1.6 of the form

$$\epsilon \leq \frac{\log_{\max(2^p, D^3)}(2)p}{(4C')^{\frac{p}{p-1}}(4D)^{\frac{4}{p-1}} + \log_{\max(2^p, D^3)}(2)p} p. \tag{4.32}$$

This may be simplified if $2^p \geq D^3$, in which case we obtain the bound

$$\epsilon \leq \frac{1}{(4C')^{\frac{p}{p-1}}(4D)^{\frac{4}{p-1}} + 1} p.$$

As p further increases, the asymptotic behavior of this expression is $\frac{p}{4C'+1}$. This seemingly looses dependence on the doubling constant D. However, Theorem 1.5 gives that the A_p-connectivity constant is related to both C_{PI} and D. More precicely, by using similar techniques to [11] and the arguments in Theorem 1.5, we can show that $C' \leq 2^6 D^3 C_{PI}$ suffices, which gives the following bound for Theorem 1.4 (when $2^p \geq D^3$)

$$\epsilon \leq \frac{p}{(2^8 D^3 C_{PI})^{\frac{p}{p-1}} (4D)^{\frac{4}{p-1}} + 1},$$

which is a better bound than above.

Combining the results of this section we obtain the following characterization result for Poincaré inequalities when $p > 1$.

Theorem 4.33. *For a proper metric measure space (X, d, μ) the following conditions are equivalent.*

(1) *X satisfies a $(1, p)$-Poincaré inequality and is D-doubling for some $D > 0$.*
(2) *X is A_p-connected.*
(3) *X is q-max-connected for some $q < p$.*

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