

CHARACTERIZING SPACES SATISFYING POINCARÉ INEQUALITIES AND APPLICATIONS TO DIFFERENTIABILITY

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ABSTRACT. We show that a proper metric measure space is a RNP-differentiability space if and only if it is rectifiable in terms of doubling metric measure spaces with some Poincaré inequality. This result characterizes metric measure spaces that can be covered by spaces admitting Poincaré inequalities, as well as metric measure spaces that admit a measurable differentiable structure which permits differentiation of Lipschitz functions with certain Banach space targets. The proof is based on a new “thickening” construction, which can be used to enlarge subsets into spaces admitting Poincaré inequalities. We also introduce a new characterization in terms of a quantitative connectivity condition of spaces admitting some local $(1, p)$ -Poincaré inequality. This characterization result has several applications of independent interest. We resolve a question of Tapio Rajala on the existence of Poincaré inequalities for the class of $MCP(K, n)$ -spaces which satisfy a weak Ricci-bound. We show that deforming a geodesic metric measure space by Muckenhoupt weights preserves the property of possessing a Poincaré inequality. Finally, the new condition allows us to strengthen the conclusion of the celebrated theorem by Keith and Zhong to show that many classes of weak Poincaré inequalities self-improve to true Poincaré inequalities.

CONTENTS

1.	Introduction	1
2.	Notational conventions and preliminary results	12
3.	Proving Poincaré inequalities	17
4.	Corollaries and Applications	30
5.	PI-rectifiability, asymptotic connectivity and differentiability	32
	Appendix A. Tangents of RNP-Lipschitz differentiability spaces	55
	References	57

1. INTRODUCTION

1.1. **Overview.** This paper focuses on the problem of characterizing geometrically two analytic conditions on a metric space: the existence of a certain measurable differentiable structure and the property of possessing a Poincaré inequality. Since they were introduced, many fundamental questions about the relationships and the geometric nature of these concepts have remained open.

Poincaré inequalities for metric spaces were introduced by Heinonen and Koskela in [38], and have been central tools in the study of such concepts as Sobolev spaces and quasiconformal maps in metric spaces. Spaces satisfying such inequalities, and

which are measure doubling, are called PI-spaces. For precise definitions see Definition 2.13 and 2.11. While characterizations of a subclass of PI-spaces appeared in [38], and a general characterization in [43], it remained a question to find constructions of these spaces and flexible ways of proving them in particular contexts. For example, see the detailed discussion in [37].

Measurable differentiable structures were introduced by Cheeger in [18]. His main theorem showed that a PI-space possesses a measurable differentiable structure, which permits differentiation of Lipschitz almost everywhere. The spaces satisfying such a theorem, without the PI-space assumption, are called *differentiability spaces* (or Lipschitz differentiability spaces [44]). See below at Definition 5.2 and 5.3.

An early question was if the assumptions of Cheeger were in some sense necessary [37]. Because positive measure subsets of PI-spaces may be totally disconnected while they remain differentiability spaces [9], strictly speaking, a Poincaré inequality can not be necessary. However, it remained a question if a PI-space structure could be recovered in a weaker form such as by taking tangents or by covering the space in some form. This question was related to better understanding the local geometry of PI-spaces. Various authors, such as Heinonen, Kleiner, Cheeger and Schioppa, posed similar questions. A strong form of this question appeared in [24], where it was asked if every differentiability space is PI-rectifiable. Call a metric space *PI-rectifiable* if it can be covered up to measure zero by a countable number of subsets of PI-spaces. See Definition 5.1.

Our main result fully resolves the PI-rectifiability question for a subclass of RNP-differentiability spaces. Conversely, together with a result of Cheeger and Kleiner [22], it fully characterizes PI-rectifiable spaces.

Theorem 1.1. *A complete metric measure space (X, d, μ) is a RNP-Lipschitz differentiability space if and only if it is PI-rectifiable and every porous set has measure zero.¹*

RNP-differentiability is *a priori* a stronger assumption than Lipschitz differentiability. In a RNP-differentiability space one can differentiate all Lipschitz functions with values in RNP-Banach spaces, instead of just ones with finite dimensional targets. These spaces were introduced in the pioneering work of Li and Bate [8], where they showed that such differentiability spaces satisfied certain asymptotic and non-homogeneous versions of Poincaré inequalities. Also, an earlier paper by Cheeger and Kleiner [22] showed that every PI-space was a RNP-differentiability space.

Recent examples by Andrea Schioppa in [4] show that, indeed, there exist differentiability spaces which are not RNP-differentiability spaces. Or equivalently (by our result), they are not PI-rectifiable. Schioppa's and our work together demonstrate that differentiability of Lipschitz functions depends on the target, and that a sufficiently strong assumption of differentiability is equivalent to possessing a Poincaré inequality in some sense. This work exposes an interesting problem of understanding how this dependence on the target is related to the local geometry of the space.

¹The assumption of porous sets is somewhat technical and is similar to the discussion in [7].

In fact, one might reasonably suspect that the ability to differentiate scalar Lipschitz functions implies the ability to differentiate Lipschitz functions with infinite dimensional targets. For example, considering sequence spaces such as l_p , one might attempt to differentiate each component separately and argue that they combine to a derivative of the whole function. However, to execute this, which was done in [22], one needs the ability to express the total derivative using directional derivatives along curve fragments, and then to argue that pairs of points are well-connected by such curves. Together with the work of Schioppa, our paper shows that this argument can not be completed without assuming PI-rectifiability, and thus the paper [22] is almost optimal.

The proof of Theorem 1.1 rests on two contributions that are of independent interest. Our starting point is the work of Bate and Li [8], where they identified a decomposition of the space into pieces with asymptotic non-homogeneous Poincaré inequalities. For us, it is more important that these subsets satisfy a quantitative connectivity condition. In order to prove PI-rectifiability, we need to be able to enlarge, or “thicken” these possibly totally disconnected subsets into spaces with better connectivity properties. See Theorem 1.18 below for a more detailed discussion.

Once this enlarged space is constructed, one needs to verify that it satisfies a Poincaré inequality. This required a way of identifying a quantitative connectivity condition that is easier to establish on the space, and showing that this connectivity condition is equivalent to a Poincaré inequality. In other words, we needed a new and weaker characterization of Poincaré inequalities. We do this by introducing (in Definition 1.8) a novel condition on a metric measure space that we call (C, δ, ϵ) -connectivity. An interesting feature of this condition is that it is formally very similar to the definition of Muckenhoupt weights, and thus our methods draw a close formal similarity between the theories of Poincaré inequalities and the theory of Muckenhoupt weights. This similarity has already been observed in some ways in self-improvement phenomena by Keith and Zhong [45].

In terms of this condition, we show the following.

Theorem 1.2. *A (D, r_0) -doubling complete metric measure space (X, d, μ) admits a local $(1, p)$ -Poincaré inequality for some $1 \leq p < \infty$ if and only if it is locally (C, δ, ϵ) -connected for some $0 < \delta, \epsilon < 1$. Both directions of the theorem are quantitative in the respective parameters.*

The p in the above theorem is the exponent in the Poincaré inequality, which measures the quality of the inequality. A larger p means worse connectivity. Notable from our perspective is that the characterization is for a $(1, p)$ -Poincaré inequality for some p , and that our characterization applies for any doubling metric measure space. Previous characterizations either assumed Ahlfors regularity [38], or presumed knowledge of the exponent p [43]. As demonstrated by examples of Schioppa [58], it is possible for this exponent to be arbitrarily large, and thus applying characterizations of [43] seem difficult. Further, our formalism avoids modulus estimates, and seems easier to apply in our context.

The new characterization of Poincaré inequalities has several applications of independent interests. We present three of the most notable in this paper.

The first answers to the positive a question of Tapio Rajala on existence of Poincaré inequalities on certain metric measure spaces with weak synthetic Ricci curvature bounds. These spaces, called $MCP(K, n)$ -spaces, were introduced by

Ohta in [52]. We show that, at least for a large enough exponent, these spaces satisfy a Poincaré inequality. One expects that the exponent could be chosen much smaller.

Theorem 1.3. *If (X, d, μ) is a MCP(K, n) space, then it satisfies a local $(1, p)$ -Poincaré-inequality for $p > n + 1$. Further, if $K \geq 0$, then it satisfies a global $(1, p)$ -Poincaré inequality.*

We next consider more general self-improvement phenomena for Poincaré inequalities. In a celebrated paper, Keith and Zhong proved that in a doubling complete metric measure space a $(1, p)$ -Poincaré-inequality immediately improves to a $(1, p - \epsilon)$ -Poincaré inequality [45]. We ask if such self-improvement can be pushed to show that more general Poincaré-type inequalities also self-improve.

By a Poincaré type-inequality we will refer to inequalities that control the oscillation of a function by some, possibly non-linear, functional of its gradient. Such inequalities have appeared in the work of Semmes [61], Bate and Li [8] and Jana Björn in [10]. We will here show that on quasiconvex doubling metric measure spaces many types of Poincaré inequalities imply a true Poincaré inequality. In particular, most of the definitions of Poincaré inequalities produce the same category of PI-spaces.

Theorem 1.4. *Suppose that (X, d, μ) is a quasiconvex D -measure doubling metric measure space and satisfies a non-homogeneous (Φ, Ψ, C) -Poincaré inequality. Then (X, d, μ) has a true $(1, q)$ -Poincaré inequality for some $q > 1$. All the variables are quantitative in the parameters.*

In particular, the non-homogeneous Poincaré inequalities considered by Bate and Li in [8] improve to true Poincaré inequalities.

Remark: One could weaken the assumption of quasiconvexity if we allow any function f instead of Lipschitz functions, or by modifying the inequality to the ones considered in [8, 28]. We also obtain the following result strengthening the conclusion of Dejarnette [27].

Theorem 1.5. *Suppose (X, d, μ) is a doubling metric measure space satisfying a weak $(1, \Phi)$ -Orlicz-Poincaré inequality in the sense of [10], then it satisfies also a $(1, q)$ -Poincaré inequality for some q .*

Remark: By repeating the argument from Cheeger in [18] we can conclude that any space with a non-homogeneous Poincaré inequality is quasiconvex.

Most notably, the property of being a PI-space can be recovered even if the function Φ decays arbitrarily fast at the origin. The exponent will grow in such cases, but one can not fully lose a Poincaré inequality by such examples.

Finally, we will show a theorem concerning A_∞ -weights on metric measure spaces. These weights can vanish and blow-up on “small” subsets of the space, and thus allow flexibility in obtaining weighted Poincaré inequalities. This generalizes some aspects from [33] concerning sub-Riemannian metrics and vector fields satisfying the Hörmander condition. For the definition, see section 4 or Definition 1.16.

Theorem 1.6. *Let (X, d, μ) be a geodesic D -measure doubling metric measure space. If (X, d, μ) satisfies a $(1, p)$ -Poincaré inequality and $\nu \in A_\infty(\mu)$, then there is some $q > 1$ such that (X, d, ν) -satisfies a $(1, q)$ -Poincaré inequality.*

In the following subsections, we present the above theorems in more detail and explain some historical connections. For general exposition of related ideas and concepts, see the expository articles [11, 37, 46, 63].

All the results in this paper will be stated for proper complete metric measure spaces (X, d, μ) equipped with a Radon measure μ . Most of the results could also be modified to apply to non-complete spaces.

1.2. Characterizing spaces with Poincaré inequalities. Poincaré inequalities and doubling measures are a useful tool in analysis of differentiable manifolds and metric spaces alike (see Section 2 for the definitions). Following the convention established by Cheeger and Kleiner a space with both a doubling measure and a Poincaré inequality is referred to as a *PI-space* or *p-Poincaré space* (see Definition 2.13 below). For a general overview of these spaces and some later developments, see the beautiful expository article by Heinonen [37] and the book [39].

Due to the many desirable structural properties that PI-spaces have, much effort has been expended to understand what geometric properties guarantee a Poincaré inequality in some form. By now, several classes of spaces with Poincaré inequalities are known. See for example [16, 19, 23, 40, 43, 48, 50, 55, 58, 59]. The proofs that these examples satisfy Poincaré inequalities are often challenging, and make extensive use of the geometry of the underlying space. Our first theorem is thus motivated by the following question, which has also been posed in [37].

Question 1.7. Which geometric properties characterize PI-spaces? Find new and weak conditions on a space that guarantee a Poincaré inequalities.

As an answer to the above question, we introduce the following connectivity, or avoidance, property for a general metric measure space.

Definition 1.8. Let² $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x, y) \in X \times X$ is a pair of points with $d(x, y) = r > 0$, then we say that the pair (x, y) is (C, δ, ϵ) -connected if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr))$ there exists a 1-Lipschitz³ curve fragment $\gamma: K \rightarrow X$ connecting x and y , such that the following hold.

- (1) $\text{len}(\gamma) \leq Cd(x, y)$
- (2) $|\text{Undef}(\gamma)| < \delta d(x, y)$
- (3) $\gamma^{-1}(E) \subset \{0, \max(K)\}$

We call (X, d, μ) a $(C, \delta, \epsilon, r_0)$ -connected space, if every pair of points $(x, y) \in X$ with $r = d(x, y) \leq r_0$ is (C, δ, ϵ) -connected.

We say that X is uniformly $(C, \delta, \epsilon, r_0)$ -connected along S , if every $(x, y) \in X$ with $x \in S$ and with $d(x, y) \leq r_0$ is (C, δ, ϵ) -connected in X .

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$ -connected for all r_0 , we simply say that (X, d, μ) is (C, δ, ϵ) -connected.

This condition is very similar to the condition appearing in the work of Sean Li and David Bate [8, Theorem 3.8]. The main difference is that Li and Bate treat this property only in connection to certain classes of differentiability spaces, while

²The definition is only interesting for $\delta < 1$. However, we allow it to be larger to simplify some arguments below. If $\delta \geq 1$ then the curve fragment could consist of two points $\{0, d(x, y)\}$, and $\gamma(0) = x, \gamma(d(x, y)) = y$.

³According to Lemma 2.15 we could simply assume some Lipschitz bound, but it is easier to normalize the situation.

here we isolate it as a property of a general metric measure space (see below for more discussion). Another related condition appears in connection to ∞ -Poincaré inequality discussed in [29, Theorem 3.1(f)], but there one must assume $\mu(E) = 0$.

The notion of a curve fragment and related terminology is introduced in Definition 2.14 below. For certain purposes we could also use curves γ , and replace the second and third condition by

$$\int_{\gamma} 1_E ds \leq \delta d(x, y).$$

However, while less intuitive to state with curve fragments, that language is necessary for the application to differentiability spaces in section 5, since the spaces we construct may be *a priori* disconnected. Also, it is often easier to construct curve fragments than curves, since they permits certain jumps.

Definitions involving “curves with jumps” have appeared earlier in [13, 18, 38]. The difference between these notions is that the domain of a curve fragment may be totally disconnected instead of a union of intervals. The idea of using curve fragments to study differentiability spaces is from Bate [6].

Our connectivity condition is formally very similar to the definition of Muckenhoupt weights A_{∞} in terms of a quantitative version of absolute continuity. For definitions and a beautiful exposition of this theory consult [64, Section V]. Consequently, many of our arguments and theorems are motivated by the corresponding proofs for Muckenhoupt weights.

We present some general results for this new notion. For example, it is stable under Gromov-Hausdorff limits.

Theorem 1.9. *If $(X_i, d_i, \mu_i, x_i) \rightarrow (X, d, \mu, x)$ is a convergent sequence of proper pointed metric measure spaces in the measured Gromov-Hausdorff sense, and each X_i is (C, δ, ϵ) -connected, then also the limit space (X, d, μ) is (C, δ', ϵ) -connected for every $\delta' > \delta$.*

In terms of this new notion of connectivity we prove the following.

Theorem 1.2. *A (D, r_0) -doubling complete metric measure space (X, d, μ) admits a local $(1, p)$ -Poincaré inequality for some $1 \leq p < \infty$ if and only if it is locally (C, δ, ϵ) -connected for some $0 < \delta, \epsilon < 1$. Both directions of the theorem are quantitative in the respective parameters.*

Note, the doubling is not needed for the converse statement, because by Lemma 3.2 it is implied by this condition. Similarly, the theorems below could dispense with the assumption of D -doubling. However, we add this to explicate the dependence of the parameters on the implied doubling constant.

The previous theorem fully characterizes PI-spaces. Another similar characterization of spaces with Poincaré inequalities appears in the work of Keith [43], but there the characterization is for a fixed p , and in terms of modulus estimates. In some cases, one might be interested in Poincaré inequalities without knowing *a priori* the value of p sought. Also, in general p might be arbitrarily large [59]. In such contexts, our characterization seems easier to apply.

Other characterizations in terms of Loewner properties appeared earlier in the paper of Heinonen and Koskela [38], but there one had to assume Ahlfors regularity. Compared to these results, our result is *a priori* weaker in that it does not specify the optimal range of p . However, it leads to new applications because the condition is easier to obtain in some contexts, and we don't need Ahlfors regularity. This is

demonstrated in the results below, where we apply this theorem to prove Poincaré inequalities for new classes of spaces. We remark that Keith's characterization also has several interesting applications [50, 59] and our theorem doesn't replace it.

We introduce a finer notion of connectivity in terms of which the main theorem becomes easier to state. It should be compared to the relationship between A_∞ -weights and A_p -weights in classical analysis.

Definition 1.10. We call a metric measure space (X, d, μ) finely α -connected with parameters (C_1, C_2) if for any $0 < \tau < 1$ the space is $(C_1, C_2\tau^\alpha, \tau)$ -connected. Further, we say that the space is locally finely (α, r_0) -connected with parameters (C_1, C_2) if it is $(C_1, C_2\tau^\alpha, \tau, r_0)$ -connected for any $0 < \tau < 1$. If we do not explicate the dependence on the constants, we simply say (X, d, μ) is finely α -connected if constants C_1, C_2 exist with the previous properties. Similarly, we define locally finely (α, r_0) -connected spaces.

For example, note that \mathbb{R}^n is finely $\frac{1}{n}$ -connected. Similarly, the space arising from gluing two copies of \mathbb{R}^n along the origin is finely $\frac{1}{n}$ -connected. If the Lebesgue measure λ on \mathbb{R}^n is deformed by an A_p -Muckenhoupt weight w , then the space $(\mathbb{R}^n, |\cdot|, w\lambda)$ is $\frac{1}{pn}$ -connected. This follows from the results in [64, Chapter V].

The finer notion of connectivity is used as an intermediate step to prove Theorem thm:contheorem (see also the quantitative version in 3.16). It also allows us to better control the exponent in the Poincaré inequality. The connection between the previous two definitions is given by the following theorem.

Theorem 1.11. *Assume $0 < \delta < 1$ and $0 < \epsilon, r_0, D$. If (X, d, μ) is a $(C, \delta, \epsilon, r_0)$ -connected (D, r_0) -doubling metric measure space, then there exist $0 < \alpha, C_1, C_2$ such that it is also finely $(\alpha, r_0/(C_1 8))$ -connected with parameters (C_1, C_2) . We can choose $\alpha = \frac{\ln(\delta)}{\ln(\frac{\epsilon}{2M})}$, $C_1 = \frac{C}{1-\delta}$ and $C_2 = \frac{2M}{\epsilon}$, where $M = 2 \max(D^{-\log_2(1-\delta)+1}, D^3)$.*

Remark: We remark, that while the connectivity conditions are very similar to Muckenhoupt conditions, a major difference holds. Both conditions admit a “self-improvement” property of the form in Theorem 1.11. This can be thought of as the statement that A_∞ -weights are A_p -weights for some p . However, Muckenhoupt weights and Poincaré inequalities also possess a different type of self-improvement. An A_p -weight automatically belongs to $A_{p-\epsilon}$ for some $\epsilon > 0$, and a $(1, p)$ -Poincaré inequality on a doubling metric measure space improves to a $(1, p - \epsilon)$ -Poincaré inequality (see [45]). However, a finely α -connected space may fail to be $\alpha + \epsilon$ -connected for any ϵ . For example, \mathbb{R}^n is finely $\frac{1}{n}$ -connected but not finely α -connected for any $\alpha > \frac{1}{n}$. Thus, fine connectivity is not an open condition.

Finally, the last step in the proof of Theorem 1.2 is the following. See also Theorem 3.10 below.

Theorem 1.12. *Let (X, d, μ) be a D -measure doubling locally finely (α, r_0) -connected metric measure space with parameters (C_1, C_2) , then for any $p > \frac{1}{\alpha}$ the space satisfies a local $(1, p, r_0/(8C_1), C_{PI}, 2C_1)$ -Poincaré inequality. In short, the space is a PI-space.*

We can set $M = 2(C_2 D^4)^{\frac{1}{p\alpha-1}}$, $\delta = C_2 \left(\frac{D^4}{M^p}\right)^\alpha$, $C_3 = \frac{C_1 M}{1-M\delta}$ and $C_{PI} = 2C_3$.

The range of p 's in this theorem is tight in general. Take the space Y arising as the gluing of two copies of \mathbb{R}^n through their origins. The resulting space (Y, d, μ) , where d is the glued metric and μ the sum of the measures on each component, is

$\frac{1}{n}$ -connected and satisfies a $(1, p)$ -Poincaré inequality only for $p > n$. On the other hand, for some particular examples, such as \mathbb{R}^n or Ricci-bounded manifolds [19], we know that the space satisfies a $(1, 1)$ -Poincaré inequality, but are only $\frac{1}{n}$ -connected. Also, if this result is applied to Muckenhoupt weights $w \in A_q(\mathbb{R}^n)$, we would get a Poincaré inequality for $p > nq$, while it is known to hold for $p > q$ (see e.g. [64, Chapter V] or [30]). Finally, we remark that this theorem is not an equivalence. A $(1, p)$ -PI-space might not be α -connected for any $\alpha \geq \frac{1}{p}$.

The proofs of the three main theorems are all very similar. They are based on an iterated gap-filling and limiting argument. Recall, that the curve fragments guaranteed by Definitions 1.8 and 1.10 can contain gaps. However, the core idea is to re-use the connectivity condition at the scale of these gaps and to replace them with finer curve fragments. A similar argument appears in [8, Lemma 3.6], and earlier in [38]. This procedure can be used quite easily to prove quasicovexity. In that case, we can always set the obstacle $E = \emptyset$ when we apply the connectivity definition. However, for the application to Theorems 1.11 and 1.12, we will need to define obstacles at different scales. To do this successfully, we use a maximal function estimate. Such iterative arguments employing maximal functions resemble the proofs of Theorems V.3.1.4 and V.5.1.3 in [64, Section V]).

Finally, while our condition and conclusion do not use the modulus estimates of Keith [43] and Heinonen-Koskela [38], it is not surprising that at the end we are able to connect our condition to a certain type of modulus estimate. To state it, we need the following terminology. Define the set of curves

$$\Gamma_{x,y,C} = \{\gamma: [0, 1] \rightarrow X \mid \gamma(0) = x, \gamma(1) = y, \text{len}(\gamma) \leq Cd(x, y)\}.$$

We call a non-negative measurable function ρ admissible for the family $\Gamma_{x,y,C}$ if for any rectifiable $\gamma \in \Gamma_{x,y,C}$ we have

$$\int_{\gamma} \rho \, ds \geq 1.$$

The p -modulus of the curve family (centered at x , and scale $s > d(x, y)$) is then defined as

$$\text{Mod}_p^{x,s}(\Gamma_{x,y,C}) = \inf_{\rho \text{ is admissible for } \Gamma_{x,y,C}} \int_{B(x,s)} \rho^p \, d\mu.$$

Theorem 1.13. *If (X, d, μ) is a complete (D, r_0) -measure doubling metric measure space which is also locally finely (α, r_0) -connected with parameters (C_1, C_2) , then for any $x, y \in X$ with $d(x, y) = r < r_0/(8C_1)$, any $p > \frac{1}{\alpha}$ we have*

$$\text{Mod}_p^{x,2C_1r}(\Gamma_{x,y,C_3}) \geq \frac{C_3^p}{r^{p-1}},$$

where C_3 is as in Lemma 3.10.

For a more detailed discussion on modulus, see [67].

1.3. Applications of the Characterization. Our new Definition 1.8 can be used in a variety of contexts to establish true Poincaré inequalities under *a priori* weaker connectivity properties.

The first result concerns metric spaces with weak Ricci bounds. These spaces originally arose following the work of Cheeger and Colding on Ricci limits [19, 20].

Many different definitions appeared such as different definitions for $CD(K, n)$ [49, 65, 66], $CD^*(K, n)$ [5] and a strengthening $RCD(K, n)$ [1, 3]. Ohta also defined a very weak form of a Ricci bound by the *measure contraction property* and introduced $MCP(K, N)$ -spaces [52]. Spaces satisfying one of the stronger conditions ($CD(K, N)$, $RCD(K, N)$ or $RCD(K, N)^*$) were all known to admit $(1, 1)$ -Poincaré inequalities [54]. It was also known that a non-branching $MCP(K, N)$ -space would admit a $(1, 1)$ -Poincaré inequality. This was observed by Renesse [68], whose proof was essentially a repetition of classical arguments in [19]. However, Rajala had conjectured that this assumption of non-branching was inessential. $MCP(K, n)$ spaces are interesting partly because they are known to include some very non-Euclidean geometries such as the Heisenberg group and certain Carnot-groups [41, 56]. We are able to prove the following.

Theorem 1.3. *If (X, d, μ) is a $MCP(K, n)$ space, then it satisfies a local $(1, p)$ -Poincaré-inequality for $p > n + 1$. Further, if $K \geq 0$, then it satisfies a global $(1, p)$ -Poincaré inequality.*

We are left with the following open problem.

Open Question: Does every $MCP(K, n)$ -space admit a local $(1, 1)$ -Poincaré inequality?

We next consider self-improvement phenomena for Poincaré inequalities. Keith and Zhong proved that in a doubling complete metric measure space a $(1, p)$ -Poincaré-inequality immediately improves to a $(1, p - \epsilon)$ -Poincaré inequality [45]. We ask if more general Poincaré-type inequalities also self-improve.

By a Poincaré type-inequality we will refer to inequalities that control the oscillation of a function by some, possibly non-linear, functional of its gradient. First, recall the definition of an upper gradient for metric measure spaces.

Definition 1.14. Let (X, d, μ) be a metric measure space and $f: X \rightarrow \mathbb{R}$ a Lipschitz function. We call a non-negative Borel-measurable g an upper gradient for f if for every rectifiable curve $\gamma: [0, L] \rightarrow X$ we have

$$|f(\gamma(0)) - f(\gamma(L))| \leq \int_0^L g(\gamma(t)) ds_\gamma.$$

With this definition, we can define a non-homogeneous Poincaré inequality.

Definition 1.15. Let (X, d, μ) be a metric measure space. Let $\Phi, \Psi: [0, \infty) \rightarrow [0, \infty)$ be increasing functions with the following properties.

- $\lim_{t \rightarrow 0} \Phi(t) = \Phi(0) = 0$
- $\lim_{t \rightarrow 0} \Psi(t) = \Psi(0) = 0$

We say that (X, d, μ) satisfies a non-homogeneous (Φ, Ψ, C) -Poincaré inequality if for every 2-Lipschitz⁴ function $f: X \rightarrow \mathbb{R}$, every upper gradient $g: X \rightarrow \mathbb{R}$ and every ball $B(x, r) \subset X$ we have

$$\int_{B(x, r)} |f - f_{B(x, r)}| d\mu \leq r\Psi \left(\int_{B(x, Cr)} \Phi \circ g d\mu \right).$$

⁴The constant 2 is only used to simplify arguments below. Any fixed bound could be used. Also, by adjusting the right-hand side by scaling with the Lipschitz constant of f , the inequality could be made homogeneous.

This class of Poincaré inequalities subsumes the ones considered by Björn in [10] and the Non-homogeneous Poincaré Inequalities (NPI) considered by Bate and Li in [8]. We get the following theorem and corollary.

Theorem 1.4. *Suppose that (X, d, μ) is D -measure doubling metric measure space and satisfies a non-homogeneous (Φ, Ψ, C) -Poincaré inequality. Then (X, d, μ) has a true $(1, q)$ -Poincaré inequality for some $q > 1$. All the variables are quantitative in the parameters.*

As already discussed in sub-section 1.1, we can similarly obtain results for Orlicz-Poincaré inequalities.

We observed a formal similarity between Definition 1.8 and the definition of Muckenhoupt weights. For our purposes we define Muckenhoupt weights as follows.

Definition 1.16. Let (X, d, μ) be a D -measure doubling metric measure space. We say that a Radon measure ν is a generalized $A_\infty(\mu)$ -measure, or $\nu \in A_\infty(\mu)$, if $\nu = w\mu$, and there exist $0 < \epsilon < 1, 0 < \delta < 1$ such that for any $B(x, r)$ and any Borel-set $E \subset B(x, r)$

$$\nu(E) \leq \delta \nu(B(x, r)) \implies \mu(E) \leq \epsilon \mu(B(x, r)).$$

If $w\mu \in A_\infty(\mu)$ for some locally integrable w , then we call w a Muckenhoupt-weight.

There are different definitions in the literature. These variants and their equivalence is discussed in [42]. For geodesic metric measure spaces many of them are equivalent. Also, notable is the class of strong A_∞ -weights introduced by David and Semmes in [26] (see also [60, 62]). They are somewhat different from Muckenhoupt weights, and usually form a sub-class of them [42].

For an unfamiliar reader, we remind that Muckenhoupt weights may vanish and blow-up on subsets of the space. See [64, Section V] for examples. They are also somewhat flexible to construct, as alluded to in [32, Section 3.18]. Interestingly enough, deformations by such weights preserve on geodesic spaces the property of possessing a Poincaré inequality.

Theorem 1.6. *Let (X, d, μ) be a geodesic D -measure doubling metric measure space. If (X, d, μ) satisfies a $(1, p)$ -Poincaré inequality and $\nu \in A_\infty(\mu)$, then there is some $q > 1$ such that (X, d, ν) -satisfies a $(1, q)$ -Poincaré inequality.*

1.4. Relationship between differentiability spaces and PI-spaces. Cheeger [18] defined a metric measure analog of a differentiable structure and proved a powerful generalization of Rademacher's theorem for PI-spaces. The spaces admitting differentiation are here called *differentiability spaces*. See below at Definition 5.2 and 5.3. Our main result is that a subclass of these spaces, called RNP-differentiability spaces, are PI-rectifiable (see 5.1 for a definition). Thus, within this subclass the conditions of Cheeger are both necessary and sufficient.

Theorem 1.1. *A complete metric measure space (X, d, μ) is a RNP-Lipschitz differentiability space if and only if it is PI-rectifiable and every porous set has measure zero.*⁵

This result builds heavily on earlier work by Bate and Li [8], where the authors noticed that RNP-differentiability spaces satisfy certain asymptotic and non-homogeneous Poincaré inequalities. Even earlier, a number of similarities between

⁵The assumption of porous sets is somewhat technical and is similar to the discussion in [7].

PI-spaces and Poincaré inequalities were discovered: asymptotic doubling [9], an asymptotic lip-Lip equality almost everywhere [6,57], large family of curve fragments representing the measure and “lines” in the tangents [24]. These works developed ideas and gave strong support for some theorem like 1.1.

Sean Li and David Bate initiated the detailed study of RNP-differentiability spaces, which permit differentiation of Lipschitz functions with values in RNP-Banach spaces (*Radon Nikodym Property*). For more information on such Banach spaces see [53]. The question of differentiability of RNP-Banach space valued Lipschitz functions arose in the work of Kleiner and Cheeger where it was shown that a PI-space admits a Rademacher theorem for such functions [22]. Li and Bate studied the abstract condition of being an RNP-differentiability space and managed to prove a partial converse by showing that the tangents of RNP-differentiability spaces admit a weak form of Poincaré inequality and that the spaces themselves admit a form of non-homogeneous Poincaré inequality infinitesimally. This already showed that differentiability was closely connected to Poincaré-type inequalities. However, it left open a question, if differentiability spaces were PI-rectifiable, i.e. if they can be covered by positive measure subsets of PI-spaces. This question is resolved here. Further, as a consequence we obtain the following theorem, which has an easier proof presented in the Appendix to this paper A.

Theorem 1.17. *Let (X, d, μ) be an RNP-differentiability space. Then X can be covered by positive measure subsets V_i , such that each V_i is metric doubling, when equipped with its restricted distance, and for μ -a.e. $x \in V_i$ each space $M \in T_x(V_i)$ admits a PI-inequality of type $(1, p)$ for all p sufficiently large.*⁶

Our proof of Theorem 1.1 starts off with citing the result of Bate and Li that decomposes a RNP-differentiability space (X, d, μ) into parts with asymptotic and non-homogeneous forms of Poincaré inequalities, as well as a uniform and asymptotic form of Definition 1.8. We introduce two new ideas to use these pieces. On the one hand, we observe that our connectivity condition implies a Poincaré inequality using Theorem 3.16, and this already gives a Poincaré inequality for tangents of RNP-differentiability spaces (see the appendix A). To obtain PI-rectifiability we also need the ability to enlarge the pieces used by Bate and Li to satisfy, intrinsically, the connectivity property in 1.8.

The following theorem is used to construct the enlarged space. The terminology used is defined later in 5 and 2. A crucial observation is that the assumptions involve a relative form of doubling and connectivity, and in the conclusion we construct a space with an intrinsic Poincaré inequality and doubling. Thus, it can be used even for general subsets of PI-spaces to enlarge them to a PI-spaces (the enlarged space being different from the original space).

Theorem 1.18. *Let $r_0 > 0$ be arbitrary. Assume (X, d, μ) is a metric measure space and $K \subset A \subset X$, A is measurable and K compact. Assume further that X is (D, r_0) -doubling along A , A is uniformly $(\frac{1}{2}, r_0)$ -dense in X along K , and A with the restricted measure and distance is locally $(C, 2^{-60}, \epsilon, r_0)$ -connected⁷ along K . There exists constants $\bar{C}, \bar{\epsilon}, \bar{D} > 0$, and a complete metric space \bar{K} which is \bar{D} -doubling*

⁶The subset V is equipped with the restricted measure and metric.

⁷By an iteration argument similar to Theorem 1.11, or as presented in [8], the constant 2^{-60} -could be replaced by any $0 < \delta < 1$, but this would make our proof more technical and is unnecessary.

and $(\bar{C}, \frac{1}{2}, \bar{\epsilon}, r_0 2^{-20})$ -connected, and an isometry $\iota: K \rightarrow \bar{K}$ which preserves the measure. In particular, the resulting metric measure space \bar{K} is a PI-space.

For an intuitive, and slightly imprecise, overview of the proof of this construction one can consider the case of a compact subset $K \subset \mathbb{R}^n$. Here K is well-connected when thought of as a subset of \mathbb{R}^n , but may not be intrinsically connected. To satisfy a local Poincaré-inequality the space must be locally quasiconvex. In order to make K locally quasiconvex we will glue a metric space T to it. The space T is tree-like, and its vertices correspond to a discrete approximation of K along with its neighborhood. The vertices exist at different scales, and near-by vertices are attached by edges to each other at comparable scales. By using net-points of K , and Whitney centers for a neighborhood of K , we prevent adding too many points or edges at any given scale or location. This construction is analogous to that of a hyperbolic filling [17, Section 2] (see also a more recent presentation in English [14]). Very similar ideas also appear in the work of Bonk, Bourdon and Kleiner related to problems of quasiconformal maps [12, 13, 15].

1.5. Structure of paper. We first cover some general terminology and frequently used lemmas. In the third section, we introduce our notion of connectivity and prove basic properties and finally derive Poincaré inequalities. In the fourth section, we apply the results in both new and classical settings. In the final section, we apply the results to the study of RNP-differentiability spaces and introduce the relevant concepts. In the appendix, we include a different proof that tangents of RNP-differentiability spaces are PI-spaces and that our connectivity condition is preserved under Gromov-Hausdorff-convergence.

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2. NOTATIONAL CONVENTIONS AND PRELIMINARY RESULTS

We will be studying the geometry of complete proper metric measure spaces (X, d, μ) . Where not explicitly stated, all the measures considered in this paper will be Radon measures. An open ball in a metric space X with center x and radius r will be denoted by $B(x, r)$, and if $C > 0$ we will denote by $CB(x, r) = B(x, Cr)$.

Definition 2.1. A metric measure space (X, d, μ) , such that $0 < \mu(B(x, r)) < \infty$ for all balls $B(x, r) \subset X$ is said to be (locally) (D, r_0) -doubling if for all $0 < r < r_0$ and any $x \in X$ we have

$$(2.2) \quad \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq D.$$

We say that the space is D -doubling if this property holds for every $r_0 > 0$. Further we simply call a space doubling if there is a constant D such that it is D -doubling, and locally doubling if there are constants (D, r_0) such that it is locally (D, r_0) -doubling.

There is also a metric notion of doubling that does not refer to the measure. Our definition is often referred to as measure doubling. However, throughout this paper we will only use this stronger version, and thus simply say doubling.

Definition 2.3. A metric measure space (X, d, μ) is said to be asymptotically doubling if for almost every $x \in X$ we have

$$(2.4) \quad \limsup_{r \rightarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty.$$

We also define a relative version of doubling for subsets.

Definition 2.5. A metric measure space (X, d, μ) is said to be uniformly (D, r_0) -doubling along $S \subset X$ if for all $0 < r < r_0$ and any $x \in S$ we have

$$(2.6) \quad \frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq D.$$

Definition 2.7. A set $S \subset X$ is called porous if there exists constants c, r_0 such that for every $x \in S$ and every $r_0 > r > 0$ there exists a $y \in B(x, r)$ such that $B(y, cr) \cap S = \emptyset$. A set S is called σ -porous if there exist porous sets S_i (with possibly different constants c_i), such that

$$S = \bigcup_i S_i.$$

Definition 2.8. Let (X, d, μ) be a metric measure space and $A \subset X$ a positive measure subset. A point $x \in A$ is called an (ϵ, r_0) -density point of A if for any $0 < r < r_0$ we have

$$1 - \epsilon \leq \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))} \leq 1.$$

We say that A is uniformly (ϵ, r_0) -dense along $S \subset A$ if every point $x \in S$ is an (ϵ, r_0) -density point of A .

A map $f: X \rightarrow Y$ between two metric spaces (X, d_X) and (Y, d_Y) is **Lipschitz** if there is a constant L such that $d_Y(f(x), f(y)) \leq Ld_X(x, y)$. We will denote by $\text{LIP } f$ the global Lipschitz constant of a Lipschitz function, and the two local Lipschitz constants

$$(2.9) \quad \text{Lip } f(x) = \limsup_{r \rightarrow 0} \sup_{y \in B(x, r)} \frac{|f(x) - f(y)|}{r}$$

and

$$(2.10) \quad \text{lip } f(x) = \liminf_{r \rightarrow 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{d(x,y)}.$$

A map $f: X \rightarrow Y$ is called bi-Lipschitz if there is a constant L such that $1/Ld_X(x,y) \leq d_Y(f(x), f(y)) \leq Ld_X(x,y)$. The smallest such constant L is called the distortion of f .

Definition 2.11. Let $p \in [1, \infty)$, $C, C' > 0$ be constants. We say that a proper metric measure space (X, d, μ) equipped with a Radon measure satisfies a $(1, p)$ -Poincaré inequality with constants (C, C') if for every $r > 0$, every $x \in X$ and every Lipschitz function $f: X \rightarrow \mathbb{R}^n$.

$$(2.12) \quad \int_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq C'r \left(\int_{B(x,Cr)} \text{lip } f^p d\mu \right)^{\frac{1}{p}}.$$

Additionally, we say that the metric measure space (X, d, μ) satisfies a local $((1, p, r_0)$ -Poincaré inequality if for some $r_0 > 0$ the previous holds for all $r < r_0$. A space satisfies a local $(1, p)$ -Poincaré inequality if the aforementioned holds for some $r_0 > 0$.

Remark: There are different versions of Poincaré inequalities and their equivalence in various contexts has been studied in [43] and [35, 36]. The quantity $\text{lip } f$ on the right-hand side could also be replaced by $\text{Lip } f$ (see equations 2.10 and 2.9 below), and on complete spaces by an upper gradient (in any of the senses discussed in [18, 28, 38]). For non-complete spaces, which we do not focus on, the issue is slightly more delicate (see counterexamples in [47]), but can often be avoided by taking completions. Further, as long as the space is complete, we do not need to constrain the inequality for Lipschitz functions but could also use appropriately defined Sobolev spaces.

Note that on the right-hand side the ball is enlarged by a factor $C \geq 1$. Some authors call inequalities with $C > 1$ “weak” Poincaré inequalities, but we do not distinguish between these different terms. On geodesic metric spaces the inequality can be improved to have $C = 1$ [35].

We use the following notion of PI-space.

Definition 2.13. A proper metric measure space (X, d, μ) equipped with a Radon measure μ is called a $(1, p, D, r_0, C_{PI}, C)$ -PI space if it is (D, r_0) -doubling and satisfies a local $(1, p, r_0)$ -Poincaré inequality with constants (C, C_{PI}) . Further a space is called a $(1, p, D, r_0)$ -PI space, a $(1, p, r_0)$ -PI space or simply PI-space if there exists remaining constants so that the space satisfies the aforementioned property.

Definition 2.14. A curve fragment in a metric space (X, d) is a Lipschitz map $\gamma: K \rightarrow X$, where $K \subset \mathbb{R}$ is compact. For simplicity, we translate the set so that $\min(K) = 0$. We say the curve fragment connects points x and y if $\gamma(0) = x, \gamma(\max(K)) = y$. Further, define $\text{Undef}(\gamma) = [0, \max(K)] \setminus K$. If $K = [0, \max(K)]$ is an interval we simply call γ a curve.

The length of a curve fragment is defined as

$$\text{len}(\gamma) = \sup_{x_1, \dots, x_n \in K} \sum_{i=1}^n d(\gamma(x_{i+1}), \gamma(x_i)).$$

Since γ is assumed to be Lipschitz we have $\text{len}(\gamma) \leq \text{LIP}(\gamma) \max(K)$.

Analogous to curves we can define an integral over a curve fragment $\gamma: K \rightarrow X$. Denote $\sigma(t) = \sup_{x_1 \leq \dots \leq x_n \in K \cap [0, t]} \sum_{i=1}^n d(\gamma(x_{i+1}), \gamma(x_i))$. The function $\sigma|_K$ is Lipschitz on K . Thus, it is differentiable for almost every density point $t \in K$ and for such t we set $d_\gamma(t) = \sigma'(t)$ and call it the *metric derivative* (see [2] and [6]). We define an integral of a Borel function g as follows

$$\int_\gamma g \, ds = \int_K g(\gamma(t)) \cdot d_\gamma(t) \, dt,$$

when the right-hand side makes sense. This is true for example if g is bounded from below or above.

We will need to do some technical modifications of domains of Lipschitz functions and as such we formulate two lemmas useful for that. First, we give a standard re-parametrization result which generalizes to curve fragment the unit-speed parametrization for curves.

Lemma 2.15. *Let (X, d) be a complete metric space, $x, y \in X$ points and $\gamma: K \rightarrow X$ a curve fragment connecting x to y . There is a compact set $K' \subset [0, \text{len}(\gamma)]$, with $0, \text{len}(\gamma) \in K'$, and a function $\phi: K' \rightarrow K$ such that $\gamma' = \gamma \circ \phi$ is a 1-Lipschitz curve fragment connecting x to y . Further, the metric derivative is given, almost everywhere, by $d_{\gamma'} = d_\gamma \circ \phi$. If $(a_i, b_i) \subset [0, \text{len}(\gamma)] \setminus K'$ is a maximal open interval in $[0, \text{len}(\gamma)] \setminus K'$, we have $d(\gamma(a_i), \gamma(b_i)) = |a_i - b_i|$, and*

$$\text{len}(\gamma') = \text{len}(\gamma).$$

Proof: Define $\sigma: K \rightarrow \mathbb{R}$ by $\sigma(t) = \sup_{x_i \in K \cap [0, t]} \sum_{i=1}^n d(\gamma(x_{i+1}), \gamma(x_i))$, where $x_i \leq x_j$ for $i \leq j$. The function $\sigma|_K$ is automatically continuous and increasing and the result follows by considering $\sigma(K) = K'$ and its right-inverse $\phi = \sigma^{-1}$. The remaining properties are easy to check. □

Sometimes gaps in a curve will need to be enlarged.

Lemma 2.16. *Let (X, d) be a complete metric space, $x, y \in X$ points and $\gamma: K \rightarrow X$ a 1-Lipschitz curve fragment. If $[0, \max(K)] \setminus K = \bigcup_i (a_i, b_i)$ where $(a_i, b_i) = I_i$ are disjoint open intervals, and $C > C_i > 1$ are constants, then there exists a compact K' , a continuous and strictly increasing function $\phi: [0, \max(K)] \rightarrow [0, \max(K')]$ such that $\phi(K) = K'$ and $\gamma' = \gamma \circ \phi^{-1}$ defines a 1-Lipschitz curve fragment on K' . Also $[0, \max(K')] \setminus K' = \bigcup_i (\phi(a_i), \phi(b_i))$, $|\phi(a_i) - \phi(b_i)| = C_i |a_i - b_i|$ and*

$$\max(K') \leq \max(K) + (C - 1)|\text{Undef}(\gamma)|.$$

Proof: Define $\phi(t) = \int_0^t (1_K + \sum_i C_i 1_{I_i}) \, dt$. The rest of the estimates are easy to see. □

For a locally integrable function $f: X \rightarrow \mathbb{R}$ and a ball $B = B(x, r) \subset X$ with $\mu(B(x, r)) > 0$ we define the average of a function as

$$f_B = \int_B f \, d\mu = \frac{1}{\mu(B(x, r))} \int f(y) \, d\mu_y.$$

We will frequently use the Hardy-Littlewood maximal function at scale $s > 0$:

$$M_s f(x) = \sup_{0 < r < s} \int_{B(x, r)} |f|(x) \, d\mu,$$

and refer to [64] for a standard proof of the following result.

Theorem 2.17. *Let (X, d, μ) be a D -measure doubling metric measure space and $s > 0$ and $B(x, r) \subset X$ arbitrary, then for any non-negative $f \in L^1$ and $\lambda > 0$ we have*

$$\mu(\{M_s f > \lambda\} \cap B(x, r)) \leq D^3 \frac{\|f\|_{L^1(B(x, r+s))}}{\lambda}$$

and thus for any $1 < p \leq \infty$

$$\|M_s f\|_{L^p(\mu)} \leq C(D, p) \|f\|_{L^p(\mu)}.$$

Definition 2.18. A metric space (X, d) is called L -quasiconvex if for every $x, y \in X$ there exists a Lipschitz curve $\gamma: [0, 1] \rightarrow X$ such that $\gamma(0) = x, \gamma(1) = y$ and $\text{len}(\gamma) \leq Ld(x, y)$. A metric space is locally (L, r_0) -quasiconvex if the same holds for all $x, y \in X$ with $d(x, y) \leq r_0$.

Definition 2.19. A metric space (X, d) is called geodesic if for every $x, y \in X$ there exists a Lipschitz curve $\gamma: [0, 1] \rightarrow X$ such that $\gamma(0) = x, \gamma(1) = y$ and $\text{len}(\gamma) = d(x, y)$.

Any L -quasiconvex space is L -bi-Lipschitz to a geodesic space by defining a new distance

$$\bar{d}(x, y) = \inf_{\substack{\gamma: [0, 1] \rightarrow X \\ \gamma(0) = x, \gamma(1) = y}} \text{len}(\gamma).$$

Lemma 2.20. *If (X, d, μ) is a D -measure doubling metric measure space and f is a locally integrable function, and $B = B(x, r)$. Then for any $a \in \mathbb{R}$*

$$\frac{1}{2r} \int_B |f - f_B| \, d\mu \leq \frac{1}{r} \int_B |f - a| \, d\mu.$$

Proof: The result follows by twice applying the triangle inequality.

$$\frac{1}{2r} \int_B |f - f_B| \, d\mu \leq \frac{1}{2r} \int_B |f - a| + |a - f_B| \, d\mu \leq \frac{1}{r} \int_B |f - a| \, d\mu$$

□

3. PROVING POINCARÉ INEQUALITIES

We define a notion of connectivity in terms of an avoidance property. For the definition of curve fragments see Definition 2.14.

Definition 3.1. Let⁸ $0 < \delta$, $0 < \epsilon < 1$ and $C > 1$ be given. If $(x, y) \in X \times X$ is a pair of points with $d(x, y) = r > 0$, then we say that the pair (x, y) is (C, δ, ϵ) -connected if for every Borel set E such that $\mu(E \cap B(x, Cr)) < \epsilon\mu(B(x, Cr))$ there exists a 1-Lipschitz⁹ curve fragment $\gamma: K \rightarrow X$ connecting x and y , such that the following hold.

- (1) $\text{len}(\gamma) \leq Cd(x, y)$
- (2) $|\text{Undef}(\gamma)| < \delta d(x, y)$
- (3) $\gamma^{-1}(E) \subset \{0, \max(K)\}$

We call (X, d, μ) a $(C, \delta, \epsilon, r_0)$ -connected space, if every pair of points $(x, y) \in X$ with $r = d(x, y) \leq r_0$ is (C, δ, ϵ) -connected.

We say that X is (uniformly) $(C, \delta, \epsilon, r_0)$ -connected along S , if every $(x, y) \in X$ with $x \in S$ and with $d(x, y) \leq r_0$ is (C, δ, ϵ) -connected in X .

If (X, d, μ) is $(C, \delta, \epsilon, r_0)$ -connected for all r_0 , we simply say that (X, d, μ) is (C, δ, ϵ) -connected.

Remark: The set E above will often be referred to as an “obstacle”. Since we are working with Radon measures on proper metric spaces, to verify the condition we would only need to consider “test sets” E which are either all compact, or just open. For open sets this is trivial, since the measure is outer regular and any Borel obstacle E can be approximated on the outside by an open set E' . For compact sets, the argument goes via exhausting an open set by compact sets and obtaining a sequence of curve fragments.

Our first observation is that the connectivity condition already implies a doubling bound.

Lemma 3.2. *Let (X, d, μ) be a complete metric space and (C, δ, ϵ) -connected for some $C \geq 2$ and $0 < \delta < 1$, then it is also D -measure doubling for some $D > 1$.*

Proof: Fix $(x, r) \in X \times (0, \infty)$. Define $r_0 = r$. We can assume without loss of generality that $B(x, r_0/2) \neq B(x, r_0)$. In other words there, is a $y \in B(x, r_0) \setminus B(x, r_0/2)$. Let $s = d(x, y)$. Define $E = B(x, \delta s)$. We will show that $\mu(B(x, \delta s)) \geq \epsilon B(x, r)$.

If $\mu(B(x, \delta s)) < \epsilon B(x, r) < \epsilon B(x, Cs)$, then there is a 1-Lipschitz curve fragment $\gamma: K \rightarrow X$ connecting x to y with $|\text{Undef}(\gamma)| < \delta s$, and with $\gamma^{-1}(E) \subset \{0, \max(K)\}$. Now, $\gamma(0) = x$. We know that $(0, \delta s) \cap K \neq \emptyset$, because otherwise $\text{Undef}(\gamma)$ would be too big. Choose a $t \in (0, \delta s)$. But $d(\gamma(t), x) \leq t$ by the Lipschitz bound and $\gamma(t) \notin E$ by assumption. However, the Lipschitz bound gives also $\gamma(t) \in E$, which is a contradiction. Thus, we obtain the lower bound for volume.

Next, define $r_1 = \delta s \leq \delta r_0$. Inductively we can define $r_k \leq \delta^k r_0$ such that $\mu(B(x, r_k)) \geq \epsilon B(x, r_{k-1})$. This gives, by setting $k = 1 - \frac{1}{\log_2(\delta)}$, that

⁸The definition is only interesting for $\delta < 1$. However, we allow it to be larger to simplify some arguments below. If $\delta \geq 1$ then the curve fragment could consist of two points $\{0, d(x, y)\}$, and $\gamma(0) = x, \gamma(d(x, y)) = y$.

⁹According to Lemma 2.15 we could simply assume some Lipschitz bound, but it is easier to normalize the situation.

$$\mu(B(x, 1/2r)) \geq \mu(B(x, r_k)) \geq \epsilon^k \mu(B(x, r)).$$

This is the desired estimate 2.2. □

This lemma implies that the explicit doubling assumption in the theorems below is somewhat redundant. We leave it in order to explicate the dependence on the constants.

The definition of (C, δ, ϵ) -connectivity involves almost avoiding sets of a given relative size ϵ . One is led to ask if necessarily better avoidance properties hold for much smaller sets. This type of self-improvement closely resembles the improving properties of reverse Hölder inequalities and A_∞ -weights [64, Chapter V]. It turns out that since the connectivity holds at every scale a maximal function argument can be used to improve the connectivity estimate and allows us to take δ arbitrarily small. We introduce the notion of finely α -connected metric measure space.

Definition 3.3. We call a metric measure space (X, d, μ) finely α -connected with parameters (C_1, C_2) if for any $0 < \tau < 1$ the space is $(C_1, C_2\tau^\alpha, \tau)$ -connected. Further, we say that the space is locally finely (α, r_0) -connected with parameters (C_1, C_2) if it is $(C_1, C_2\tau^\alpha, \tau, r_0)$ -connected for any $0 < \tau < 1$. If we do not explicate the dependence on the constants, we simply say (X, d, μ) is finely α -connected if constants $C_1, C_2, r_0 > 0$ exist with the previous properties. Similarly, we define locally finely (α, r_0) -connected spaces.

Example: Clearly any finely (α, r_0) -connected space is also $(C, \delta, \epsilon, r_0)$ -connected for some parameters but the converse is not obvious. We next establish this converse for doubling metric measure spaces. This is done by an argument of filling in gaps in an iterative way. This will require maintaining some estimates at smaller scales which uses a maximal function type argument. Since this argument will also be used later to prove the main Theorem 3.16, we will first explain it in the simple context of proving quasiconvexity. Quasiconvexity could also be proved by techniques of Sean Li and David Bate [8].

Theorem 3.4. *If (X, d, μ) is a proper locally $(C, \delta, \epsilon, r_0)$ -connected metric measure space for some $C > 1, 0 < \delta, \epsilon < 1, r_0 > 0$, then it is locally (L, r_0) -quasiconvex for $L = \frac{C}{1-\delta}$*

Proof: Take an arbitrary pair of points $(x, y) \in X \times X$ and denote $r = d(x, y) \leq r_0$. Further, abbreviate $D_n = (C + \delta C + \delta^2 C + \dots + \delta^n C)$ and $C_1 = \frac{C}{1-\delta}$. We define for each $n \geq 0$ inductively 1-Lipschitz curve fragments $\gamma_n : K_n \rightarrow X$ with the following properties.

- (1) $\min(K_n) = 0, \max(K_n) \leq D_n d(x, y)$
- (2) $\text{len}(\gamma_n) \leq D_n d(x, y) \leq C_1 d(x, y)$
- (3) $|\text{Undef}(\gamma_n)| \leq \delta^n d(x, y)$

Since (X, d) is proper, this suffices to prove the statement by taking a sub-sequential limit¹⁰.

¹⁰A more careful argument could remove taking a sub-sequence, and further would allow for finding curves defined on full measure subsets of an interval without completeness. However, these observations are a bit tedious and distract from the main argument.

To initiate the iteration, define $K_0 = \{0, r\}$, $\gamma_0(0) = x$, $\gamma_0(r) = y$. Next, assume γ_n has been constructed, and denote $[0, \max(K_n)] = \bigcup_i (a_i, b_i)$, where (a_i, b_i) are disjoint, open and maximal intervals.

Dilation step: We will seek to patch each gap (a_i, b_i) with a new curve fragment but first need to stretch γ_n slightly. By Lemma 2.15 and Lemma 2.16 we can define a new curve fragment $\gamma'_n: K'_n \rightarrow X$ such that the following holds. Denote the maximal open intervals as $[0, \max(K'_n)] = \bigcup_i (a'_i, b'_i)$, where (a'_i, b'_i) corresponds to (a_i, b_i) via a re-parametrization. Then, we can ensure that $Cd(\gamma'_n(a'_i), \gamma'_n(b_i)) = |b'_i - a'_i|$ and further

$$(3.5) \quad \max(K'_n) \leq \max(K_n) + C\delta^n d(x, y) \leq D_{n+1}d(x, y)$$

Filling in: Next each of the gaps (a'_i, b'_i) is filled with a new curve fragment γ_i . Denote by $d_i = d(\gamma'_n(a'_i), \gamma'_n(b'_i)) = d(\gamma(a_i), \gamma(b_i)) \leq |b_i - a_i|$. By inductive hypothesis

$$(3.6) \quad \sum_i d_i \leq |\text{Undef}(\gamma_n)|\delta^n d(x, y)$$

By using (C, δ, ϵ) -connectivity with the obstacle $E = \emptyset$ and the pair of points $\gamma'_n(a'_i), \gamma'_n(b'_i)$ we find a 1-Lipschitz curve fragment $\gamma'_i: K_i \rightarrow X$ connecting $\gamma'_n(a'_i)$ to $\gamma'_n(b'_i)$ and $\text{len}(\gamma'_i) \leq Cd_i = |b'_i - a'_i|$ and $|\text{Undef}(\gamma'_i)| \leq \delta d_i$. By possibly dilating the curves a little more we can assume $\text{len}(\gamma'_i) = Cd_i = |b'_i - a'_i|$.

Next define $K_{n+1} = K'_n \bigcup (a'_i + K_i)$ and $\gamma_{n+1}(t) = \gamma'_n(t)$ for $t \in K'_n$ and $\gamma_{n+1}(t) = \gamma'_i(t - a'_i)$ for $t \in a'_i + K_i$. Then γ_{n+1} is easily seen to be a 1-Lipschitz curve fragment connecting x to y and defined on the compact set K_{n+1} . By Equation 3.5 above

$$\begin{aligned} \text{len}(\gamma_{n+1}) &\leq \max(K'_n) \\ &\leq D_{n+1}d(x, y). \end{aligned}$$

Also, we have

$$\text{Undef}(\gamma_{n+1}) = \bigcup_i (\text{Undef}(\gamma'_i) + a'_i),$$

and by Equation 3.6

$$\begin{aligned} |\text{Undef}(\gamma_{n+1})| &\leq \sum_i |\text{Undef}(\gamma'_i)| \\ &\leq \sum_i \delta d_i \leq \delta^{n+1} d(x, y) \end{aligned}$$

This completes the recursive proof. □

The Proof of the Main Theorem 3.16 is essentially the same. However, we need to first prove an improved version of connectivity. Both arguments will involve the same scheme as above, but additional care is needed in showing that the curve fragments exist at smaller scales. The proofs are structured as inductive proofs instead of recursion.

Theorem 3.7. *Assume $0 < \delta < 1, 0 < \epsilon, 0 < r_0, 1 < D$. If (X, d, μ) is a $(C, \delta, \epsilon, r_0)$ -connected (D, r_0) -doubling metric measure space, then there exist $0 < \alpha, C_1, C_2$ such that it is also finely $(\alpha, r_0/(C_1 8))$ -connected with parameters (C_1, C_2) . We can choose $\alpha = \frac{\ln(\delta)}{\ln(\frac{\epsilon}{2M})}$, $C_1 = \frac{C}{1-\delta}$ and $C_2 = \frac{2M}{\epsilon}$, where $M = 2 \max(D^{-\log_2(1-\delta)+1}, D^3)$.*

Remark: We do not need it for the proof, but always $0 < \alpha \leq 1$. This could be seen by the argument in Lemma 3.2, which can be turned around to give a lower bound for δ in terms of ϵ and the optimal doubling constant.

Proof: Define C_1, C_2, M, α as above. Throughout the proof denote by E an arbitrary open set in Definition 1.8. We don't lose any generality by assuming that the obstacles which are tested are open. The proof is an iterative construction, where at every stage gaps in a curve are filled and the curve fragments are slightly stretched. This argument could be phrased recursively but we instead phrase it using induction. For the following statement abbreviate $D_n = (C + \delta C + \delta^2 C + \dots + \delta^n C)$. Fix $M = 2 \max(D^{-\log_2(1-\delta)+1}, D^3)$.

For any $n \in \mathbb{N}$ we will show the following induction statement \mathcal{P}_n : If $x, y \in X$, $d(x, y) = r < r_0/(C_1 8)$ and

$$\mu(E \cap B(x, C_1 r)) \leq \left(\frac{\epsilon}{2M}\right)^n \mu(B(x, C_1 r)),$$

then there is a 1-Lipschitz curve fragment $\gamma: K \rightarrow X$ connecting x to y such that the following estimate hold:

- (1) $\min(K) = 0, \max(K) \leq D_n d(x, y)$
- (2) $\text{len}(\gamma) \leq D_n d(x, y) \leq C_1 d(x, y)$
- (3) $|\text{Undef}(\gamma)| \leq \delta^n d(x, y)$
- (4) $\gamma^{-1}(E) \subset \{0, \max(K)\}$.

This clearly is sufficient to establish the claim. Also, note that it results in the given value $\alpha = \frac{\ln(\delta)}{\ln(\frac{\epsilon}{2M})}$.

Case $n = 1$: Note, $M > D^{-\log_2(1-\delta)+1}$. Then we get

$$\begin{aligned} \mu(E \cap B(x, Cr)) &\leq \mu(E \cap B(x, C_1 r)) \leq \frac{\epsilon}{2M} \mu(B(x, C_1 r)) \\ &\leq \frac{\epsilon}{2M} D^{-\log_2(1-\delta)+1} \mu(B(x, Cr)) \leq \frac{\epsilon}{2} \mu(B(x, Cr)). \end{aligned}$$

We have assumed that the space is (C, δ, ϵ) -connected and thus the desired 1-Lipschitz curve fragment exists.

Induction step, assume statement for n , and prove for $n + 1$: Take an arbitrary x, y, E with the property $d(x, y) = r < r_0$ and

$$\mu(E \cap B(x, C_1 r)) \leq \left(\frac{\epsilon}{2M}\right)^{n+1} \mu(B(x, C_1 r)).$$

Define the set

$$E_M = \left\{ M(1_{E \cap B(x, C_1 r)}) \geq \left(\frac{\epsilon}{2M}\right)^n \right\} \cap B(x, C_1 r).$$

By standard doubling and maximal function estimates in Lemma 2.17, if $M \geq D^3$, we have $\mu(E_M) \leq \epsilon/2 \mu(B(x, C_1 r))$. Further by the case $n = 1$ there is a curve fragment $\gamma': K' \rightarrow X$ connecting x to y with the properties

- (1) $\min(K') = 0, \max(K') \leq D_n d(x, y)$
- (2) $\text{len}(\gamma') \leq C d(x, y)$,
- (3) $|\text{Undef}(\gamma')| \leq \delta d(x, y)$, and
- (4) $\gamma^{-1}(E_M) \subset \{0, \max(K)\}$.

The set $\text{Undef}(\gamma')$ is open and as such we can represent $\text{Undef}(\gamma') = \bigcup (a_i, b_i)$ with disjoint intervals such that $a_i, b_i \in K'$. We will seek to patch each gap (a_i, b_i) with a new curve fragment, but first need to stretch γ' slightly. This will proceed in the same two steps as before.

Dilation argument: Dilate the domain using Lemma 2.15 and Lemma 2.16. We will simplify notation by using the same symbols for the dilated curve. Then we can assume $d(\gamma'(a_i), \gamma'(b_i)) = D_n |b_i - a_i|$, $\min(K') = 0$ and

$$(3.8) \quad \max(K') \leq C d(x, y) + D_n \delta d(x, y) \leq D_{n+1} d(x, y)$$

Define $d(\gamma'(b_i), \gamma'(a_i)) = d_i$. By assumption on $\text{Undef}(\gamma')$ (before its dilation),

$$(3.9) \quad \sum_i d_i \leq \delta d(x, y)$$

Filling in: Further for each interval with $a_i \neq 0$ and $b_i \neq \max(K')$ both the endpoints $\gamma'(a_i), \gamma'(b_i)$ do not belong to E_M . If $a_i = 0$, then $\gamma'(b_i) \notin E_M$ and if $b_i = \max(K')$ then $\gamma'(a_i) \notin E_M$. This occurs only for possible terminal intervals. In any case, for one of the points $\gamma'(a_i), \gamma'(b_i) \notin E_M$. Say, $\gamma'(a_i) \notin E_M$. Then for the ball $B(\gamma'(a_i), C d_i)$ we have

$$\mu(E \cap B(\gamma'(a_i), C d_i)) \leq \left(\frac{\epsilon}{2E}\right)^n \mu(B(\gamma'(a_i), C d_i)).$$

Thus by the induction hypothesis we can define a 1-Lipschitz curve fragments $\gamma'_i: K_i \rightarrow X$ connecting $\gamma'(a_i)$ to $\gamma'(b_i)$ such that

- (1) $\text{len}(\gamma'_i) \leq D_n d_i$; and
- (2)

$$|[0, \max(K_i)] \setminus K_i| \leq \delta^n d_i.$$

- (3) $\gamma'^{-1}_i(E) \subset \{a_i, b_i\}$

Similarly, if $\gamma'(b_i) \notin E_M$. By possibly dilating the curves a little, while keeping the gaps the same size, we can assume $\text{len}(\gamma'_i) = D_n d_i$. This can be done by an argument similar to Lemma 2.16.

Next, define $K = K' \bigcup (a_i + K_i)$ and $\gamma(x) = \gamma'(x)$ for $x \in K'$ and $\gamma(x) = \gamma'_i(x - a_i)$ for $x \in a_i + K_i$. Then γ is easily seen to be a 1-Lipschitz curve fragment connecting x to y and defined on the compact set K . By Equation 3.8 above

$$\begin{aligned} \text{len}(\gamma) &\leq \max(K') = \max(K) \\ &\leq D_{n+1} d(x, y). \end{aligned}$$

Also we have

$$\text{Undef}(\gamma) = \bigcup_i (\text{Undef}(\gamma'_i) + a_i),$$

and by Equation 3.9

$$\begin{aligned} |\text{Undef}(\gamma)| &\leq \sum_i |\text{Undef}(\gamma'_i)| \\ &\leq \sum_i \delta^n d_i \leq \delta^{n+1} d(x, y) \end{aligned}$$

This completes the induction step. \square

The main application of the previous theorems is the establishment of curves, or curve fragments, that avoid a set on which we have poor control of the oscillation of the function. We will present an example of this type of argument here.

Example: Fix $M > 1$. Let (X, d, μ) be finely (α, r_0) -connected with parameters (C_1, C_2) . If $x, y \in X$,

$$\left(\int_{B(x, C_1 r)} g^p d\mu \right)^{\frac{1}{p}} < \epsilon,$$

and $d(x, y) = r < r_0$, then there exists a curve fragment $\gamma: K \rightarrow X$ connecting x to y , such that

$$\int_{\gamma} g ds \leq C_1 M \epsilon d(x, y)$$

and $|\text{Undef}(\gamma)| < \frac{C_2}{M^{p\alpha}}$. Observe the emergence of p in the size of the gaps in K . For p larger the gaps become smaller.

Proof: Apply the definition of fine (α, r_0) -connectedness to the set $E = \{g > M\epsilon\}$, which has measure at most $\frac{1}{M^p} \mu(B(x, C_1 r))$. Since $g \leq M\epsilon$ on the resulting curve fragment γ and $\text{len}(\gamma) \leq C_1 d(x, y)$ we obtain the desired integral estimate. \square

Theorem 3.10. *Assume (X, d, μ) is locally finely (α, r_0) -connected with parameters (C_1, C_2) and D -measure doubling. For any $p > \frac{1}{\alpha}$ there exists a constant C_3 with the following property. If $x, y \in X$ are points with $d(x, y) = r < r_0/(8C_1)$, and g is a positive Borel function with*

$$\left(\int_{B(x, 2C_1 r)} g^p d\mu \right)^{\frac{1}{p}} \leq 1,$$

then there exists a 1-Lipschitz curve $\gamma: [0, L] \rightarrow X$ connecting x to y , such that

$$\int_{\gamma} g ds \leq C_3 d(x, y).$$

We can set $C_3 = \frac{C_1 M}{1 - M\delta}$, where $M = 2(C_2 D^4)^{\frac{1}{p\alpha - 1}}$ and $\delta = C_2 \left(\frac{D^4}{M^p}\right)^{\alpha}$.

Remark: By possibly scaling g we can use the statement for non-unit L^p -bounds for g .

Proof: One can approximate g from above by a lower-semi-continuous function, and then from below by an increasing sequence of continuous functions. This standard argument (see e.g. [34, Chapter 1], [38, Proposition 2.27]) allows for assuming that g is continuous. Fix $M = 2(C_2 D^4)^{\frac{1}{p\alpha-1}}$ and choose a $\delta = C_2 \left(\frac{D^4}{M^p}\right)^\alpha$. We have $\frac{1}{M} > \delta > 0$, and $M \geq 2$.

We will construct the curve by an iterative procedure depending on n . The proof is structured by an inductive statement. The procedure is the same as in the previous two proofs: a dilation argument followed by gap-filling. However, a new issue arises as we need to ensure that the integral estimate holds at the smaller scale. Abbreviate

$$D_n = (C_1 M + C_1 M(M\delta) + \cdots + C_1 M(M\delta)^{n-1}),$$

$$C_n = (C_1 + C_1(\delta) + \cdots + C_1(\delta)^{n-1})$$

and

$$D_\infty = \lim_{n \rightarrow \infty} D_n, C_\infty = \lim_{n \rightarrow \infty} C_n.$$

Induction statement: \mathcal{P}_n : Assume $v, w \in X$ are arbitrary with $d(v, w) = r < r_0/(8C_1)$ and h is any continuous function on X with

$$\left(\int_{B(v, 2C_1 r)} h^p d\mu \right)^{\frac{1}{p}} \leq 1.$$

Then, there is a 1-Lipschitz curve fragment $\gamma : K \rightarrow X$ with $\min(K) = 0, \max(K) \leq C_n d(x, y)$,

$$(3.11) \quad \int_\gamma h ds \leq D_n d(v, w),$$

$$(3.12) \quad |\text{Undef}(\gamma)| \leq \delta^n d(v, w),$$

$$(3.13) \quad \text{len}(\gamma) < C_n d(v, w)$$

and for any maximal open set $(a_i, b_i) \subset \text{Undef}\gamma$ we have $d(\gamma(a_i), \gamma(b_i)) = |b_i - a_i| = d_i$ and for at least one $z_i \in \{a_i, b_i\}$

$$(3.14) \quad \left(\int_{B(z_i, 2C_1 d_i)} h^p d\mu \right)^{\frac{1}{p}} \leq M^n.$$

Once we have shown \mathcal{P}_n for every n we can apply it with $(x, y) = (v, w)$ and $h = g$ to obtain a sequence of curve fragments γ_n with $\int_{\gamma_n} g ds \leq D_n d(x, y)$, and by taking a limit we obtain a curve fragment γ with $\text{len}(\gamma) \leq C_\infty d(x, y)$ (from 3.13) and $\int_\gamma g ds \leq D_\infty d(x, y)$ (from 3.11). Finally, from 3.12 we can conclude that $|\text{Undef}(\gamma)| = 0$, and thus γ is in fact a curve. In the rest of the proof we show by induction \mathcal{P}_n for every natural number n .

Base case $n = 1$: Define $E_M = \{M_{2C_1\delta r}h^p > M^p\}$. Then

$$\mu(E_M \cap B(v, C_1r)) \leq \frac{D^3}{M^p} \mu(B(v, 2C_1r)) \leq \frac{D^4}{M^p} \mu(B(v, C_1r))$$

by Theorem 2.17. Thus, by the fine connectivity, there is a 1-Lipschitz curve fragment connecting v to w with

$$\text{len}(\gamma) \leq C_1 d(v, w),$$

$$|\text{Undef}(\gamma)| < C_2 \frac{(D^4)^\alpha}{M^{\alpha p}},$$

and

$$\gamma(K \setminus \{0, \max(K)\}) \subset \{M_{2C_1\delta r}h \leq M\}.$$

This immediately established 3.13. Next, we prove Estimates 3.12 and 3.11.

The estimate 3.12 follows from the choice $\delta = C_2 \left(\frac{D^4}{M^p}\right)^\alpha$. By assumption, $\gamma(K \setminus \{0, \max(K)\}) \subset \{M_{2C_1r}h^p \leq M^p\}$ and h is continuous. Thus, it follows that $h \circ \gamma \leq M$ on $K \setminus \{0, \max(K)\}$. In particular, we obtain 3.11 by the simple upper bound

$$\int_\gamma h \, ds \leq \text{len}(\gamma)M \leq C_1 M d(x, y).$$

Finally, we show Estimate 3.14. Let $(a_i, b_i) \subset [0, \max(K)] \setminus K$ be an arbitrary maximal open interval. By using Lemma 2.15 we can assume $d(\gamma(a_i), \gamma(b_i)) = |b_i - a_i| = d_i$. Also, either one of a_i, b_i is not in $\gamma^{-1}(E_M)$. Denote it by z_i . The maximal function estimate $M_{2C_1\delta r}(z_i) \leq M^p$ combined with $d_i \leq \delta r$ gives

$$\left(\int_{B(z_i, 2C_1d_i)} h^p \, d\mu \right)^{\frac{1}{p}} \leq M^p.$$

Assume \mathcal{P}_n and show \mathcal{P}_{n+1} : By the case \mathcal{P}_n there exists a 1-Lipschitz curve fragment $\gamma' : K \rightarrow X$ connecting v to w such that

$$\int_{\gamma'} h \, ds \leq D_n d(v, w),$$

$$\text{len}(\gamma') \leq C_n d(v, w),$$

$$|\text{Undef}(\gamma')| \leq \delta^n d(v, w),$$

and for any maximal open set $(a_i, b_i) \subset \text{Undef}\gamma$ we have $d(\gamma(a_i), \gamma(b_i)) = |b_i - a_i| = d_i$ and at least for one $z_i \in \{a_i, b_i\}$

$$\left(\int_{B(z_i, 2C_1d_i)} h^p \, d\mu \right)^{\frac{1}{p}} \leq M^n.$$

Filling in gaps: Now similar to the argument in Lemma 1.11 we will fill in the gaps of γ' . The set $\text{Undef}(\gamma')$ is open, and we can represent $\text{Undef}(\gamma') = \bigcup (a_i, b_i)$

with disjoint intervals such that $a_i, b_i \in K'$. Define $|b_i - a_i| = d_i$. By 3.14 for one $z_i \in \{a_i, b_i\}$ we have

$$\left(\int_{B(\gamma(z_i), 2C_1 d_i)} h^p d\mu \right)^{\frac{1}{p}} \leq M^n.$$

By the base case $n = 1$ (applied to the re-scaled g/M^n and $v = \gamma(a_i), w = \gamma(b_i)$) we can define 1-Lipschitz curve fragments $\gamma'_i: K_i \rightarrow X$ connecting $\gamma'(a_i)$ to $\gamma'(b_i)$ such that

- $\int_{\gamma'_i} g ds \leq C_1 M^n d_i$, and
-

$$|[0, \max(K_i)] \setminus K_i| \leq \delta d_i.$$

- For any maximal open sets $(a_j^i, b_j^i) \subset \text{Undef} \gamma'_i$ we have $d(\gamma(a_j^i), \gamma(b_j^i)) = |a_j^i - a_j^i| = d_j^i$ and at least for one $z_j^i \in \{a_j^i, b_j^i\}$

$$M_{2C_1 d_j^i} h^p(z_j^i) \leq M^{(n+1)p}.$$

Adjustment: First, choose a parametrization by using Lemmas 2.15 and 2.16 such that $d(\gamma'(a_i), \gamma'(b_i)) = \text{len}(\gamma'_i)$. Next define $K = K' \cup (a_i + K_i)$, and define a curve fragment $\gamma: K \rightarrow X$ by $\gamma(t) = \gamma'(t)$ for $t \in K'$ and $\gamma(t) = \gamma'_i(t - a_i)$ for $t \in a_i + K_i$. Then γ is easily seen to be a 1-Lipschitz curve fragment connecting v to w . The domain K is also clearly compact. We now verify the inequality 3.11 for $n + 1$. Note that by the argument of Lemma 2.16 and continuity h combined with $\gamma'_i(K_i) \subset \{h \leq M^{n+1}\}$ we get

$$\int_{\gamma'_i} h ds \leq \text{len}(\gamma'_i) M^{n+1} \leq C_1 \delta d_i M^{n+1} d(v, w).$$

Thus adding up the individual contributions and noting that $\sum_i d_i \leq \delta^n d(v, w)$ by assumption we obtain the desired estimate.

$$\begin{aligned} \int_{\gamma} h ds &= \int_{\gamma'} h ds + \sum_i \int_{\gamma'_i} h ds \\ &\leq D_n d(v, w) + C_1 \delta^n M^{n+1} d(v, w) \\ &\leq (C_1 M + C_1 M (M\delta) + \dots + C_1 M (M\delta)^{n+1}) d(v, w). \\ &= D_{n+1} d(v, w) \end{aligned}$$

Now prove the estimate for gaps 3.12 for $n + 1$. Note that

$$\text{Undef}(\gamma) = \bigcup_i (\text{Undef}(\gamma'_i) + \phi(a_i)),$$

and as such we obtain the desired estimate

$$\begin{aligned} |\text{Undef}(\gamma)| &\leq \sum_i |\text{Undef}(\gamma'_i)| \\ &\leq \sum_i \delta^n |b_i - a_i| \leq \delta^{n+1} d(v, w) \end{aligned}$$

Finally, check the Estimate 3.14 concerning the integral average. Any maximal open interval $I \subset \text{Undef} \gamma$ is also a translate of a maximal undefined interval for

γ'_i . I.e. we can express any such interval as $I = (a_j^i, b_j^i)$ for some maximal open interval $(a_j^i, b_j^i) \subset \text{Undef}(\gamma'_i) + a_i$. The desired Estimate 3.14 is equivalent to the desired estimate for γ'_i . □

The following lemma follows by estimating the integral over the curve using similar arguments to [8].

Lemma 3.15. *Let (X, d, μ) be a metric measure space. If f is a Lipschitz function, and $\gamma: K \rightarrow X$ is a 1-Lipschitz curve fragment, then for every $a, b \in K$*

$$|f(\gamma(a)) - f(\gamma(b))| \leq \text{LIP } f |\text{Undef}(\gamma) \cap [a, b]| + \int_{\gamma} \text{Lip } f \, ds.$$

Finally, we prove the following quantitative version of Theorem 1.2, from which it follows immediately.

Theorem 3.16. *(Main Theorem) Let (X, d, μ) be a D -measure doubling locally $(C, \delta, \epsilon, r_0)$ -connected metric measure space, then there exists C_{PI}, C_1 for $\alpha = \log_{\frac{\epsilon}{2D}}(\delta)$, $p > \frac{1}{\alpha}$ and $0 < r < \frac{r_0}{8C_1}$, such that for any Lipschitz function f and any $x \in X$*

$$\int_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq C_{PI} r \left(\int_{B(x,2C_1 r)} \text{Lip } f^p \right)^{\frac{1}{p}}.$$

We can set $C_1 = \frac{C}{1-\delta}$ and $C_{PI} = 2C_3$ where C_3 is given by Lemma 3.10. In particular, the space is a PI-space.

Proof: Denote

$$A = \left(\int_{B(x,2C_1 r)} \text{Lip } f^p \right)^{\frac{1}{p}}.$$

Let $y \in B(x, r)$ be arbitrary. First use Lemma 3.10 on $h = \text{Lip } f/A$ to give a curve γ connecting x to y with

$$\int_{\gamma} \text{Lip } f \, ds \leq C_3 A r.$$

Thus, using Lemma 3.15 we get a segment inequality in terms of a maximal function

$$\int_{B(x,r)} |f(y) - f(x)| d\mu \leq C_3 r A.$$

Together with Lemma 2.20 and the choice $a = f(x)$ this completes the proof. □

Similar techniques give a modulus estimate. For any $x, y \in X$, and any C , define the collection of curves.

$$\Gamma_{x,y,C} = \{\gamma: [0, 1] \rightarrow X | \gamma(0) = x, \gamma(1) = y, \text{len}(\gamma) \leq Cd(x, y)\}$$

We call a non-negative measurable function ρ admissible for the family $\Gamma_{x,y,C}$ is for any $\gamma \in \Gamma_{x,y,C}$ we have

$$\int_{\gamma} \rho \, ds \geq 1.$$

The p -modulus of the curve family (centered at x , and scale Cr) is then defined as

$$\text{Mod}_p^{x,s}(\Gamma_{x,y,C}) = \inf_{\rho \text{ is admissible for } \Gamma_{x,y,C}} \int_{B(x,s)} \rho^p \, d\mu.$$

By the previous arguments for a finely α -connected space we can show the following lower bound for $p > \frac{1}{\alpha}$.

Theorem 1.13. *If (X, d, μ) is a complete (D, r_0) -measure doubling metric measure space which is also locally finely (α, r_0) -connected with parameters (C_1, C_2) , then for any $x, y \in X$ with $d(x, y) = r < r_0/(8C_1)$, any $p > \frac{1}{\alpha}$ we have*

$$\text{Mod}_p^{x,2C_1r}(\Gamma_{x,y,C_3}) \geq \frac{C_3^p}{r^{p-1}},$$

where C_3 is as in Lemma 3.10.

Proof: Immediate corollary of Lemma 3.10. If ρ is admissible, and

$$\int_{B(x,2C_1r)} \rho^p \, d\mu \leq \frac{1}{2C_3^p r^p},$$

then there exists a 1-Lipschitz curve γ connecting x to y of length at most C_3r and

$$\int_{\gamma} \rho \, ds < \frac{1}{C_3r} C_3r \leq 1,$$

which contradicts the admissibility of ρ . Thus, the original modulus estimate must hold. □

Remark: The results of this section are tight in terms of the range of p . We give two examples. Taking two copies of \mathbb{R}^n glued along the origin (in fact any subset) we obtain a space that is $\frac{1}{n}$ -connected and a simple analysis based on modulus estimates tells us that it only admits a $(1, p)$ -Poincaré inequality for $p > n$. Another example arises from A_s weights w on \mathbb{R} . We know that such weights are $\frac{1}{s}$ -connected [64], and thus they admit a Poincaré inequality for $p > s$. These examples suggest that the fine α -connectivity measures the stable regime of Poincaré inequalities, i.e. those p such that $(1, p)$ -inequalities hold on glued spaces or spaces with controlled density changes.

For purposes of the following theorems we define an upper gradient of a Lipschitz function [38].

Definition 3.17. Let (X, d, μ) be a metric measure space and $C \geq 1$ some constant. Let $\Phi, \Psi: [0, \infty) \rightarrow [0, \infty)$ be increasing functions with the following properties.

- $\lim_{t \rightarrow 0} \Phi(t) = \Phi(0) = 0$
- $\lim_{t \rightarrow 0} \Psi(t) = \Psi(0) = 0$

We say that (X, d, μ) satisfies a non-homogeneous (Φ, Ψ, C) -Poincaré-inequality if for every 2-Lipschitz¹¹ function $f: X \rightarrow \mathbb{R}$, every upper gradient $g: X \rightarrow \mathbb{R}$ and every ball $B(x, r) \subset X$ we have

$$\int_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq r\Psi \left(\int_{B(x,Cr)} \Phi \circ g d\mu \right).$$

We get the following theorems.

Theorem 1.4. *Suppose that (X, d, μ) is D -doubling metric measure space and satisfies a non-homogeneous (Φ, Ψ, C) -Poincaré-inequality. Then (X, d, μ) is (C, δ, ϵ) -connected and moreover admits a true $(1, q)$ -Poincaré inequality for some $q > 1$. All the variables are quantitative in the parameters.*

Proof of Theorem 1.4: By repeating an argument from the appendix in [18], we see that X must be quasiconvex. Further, by a bi-Lipschitz change of parameters, and adjusting Ψ, Φ , we can assume that the space is geodesic. Further, by adding an increasing positive function to Ψ, Φ we can assume that $\Psi(t) > 0, \Phi(t) > 0$ for $t > 0$. By weakening the assumption, we can take Ψ, Φ to be upper semi-continuous. Assume next that for any 1-Lipschitz f and g an upper gradient and any ball $B(x, r)$.

$$\int_{B(x,r)} |f - f_{B(x,r)}| d\mu \leq r\Psi \left(\int_{B(x,Cr)} \Phi(g) d\mu \right).$$

Define the right inverses $\xi(t) = \inf_{\Psi(s) \geq t} s$. Then we have $\xi(\Psi(t)) \leq t$ and thus

$$\xi \left(\frac{1}{r} \int_{B(x,r)} |f - f_{B(x,r)}| d\mu \right) \leq \int_{B(x,r)} \Phi(g) d\mu.$$

Since $\lim_{t \rightarrow 0} \Phi(t) = 0$ we can choose a function $\sigma(t)$ such that for any $0 < s \leq \sigma(t)$ we have

$$\Phi(s) \leq t.$$

We will show $(B, \frac{1}{2}, \epsilon)$ -connectivity for $0 < \epsilon$ small enough for a specific B . Our choice will be

$$B = \max \left(C, \left[\sigma \left(\xi \left(\frac{1}{20D^5} \right) / 2 \right) \right]^{-1} \right).$$

Suppose X is not $(B, \frac{1}{2}, \epsilon)$ -connected. If this were the case, we could choose $x, y \in X$ with $r = d(x, y)$, and an obstacle E such that

$$\frac{\mu(E \cap B(x, Br))}{\mu(B(x, Br))} \leq \epsilon,$$

and such that for any 1-Lipschitz curve fragment $\gamma: [0, L] \rightarrow B(x, Br)$ connecting x to y with $\text{len}(\gamma) \leq Br$ we'd have $|\gamma^{-1}(E)| > \frac{r}{2}$. Define

$$\rho(z) = \inf_{\gamma: [0, L] \rightarrow X} \frac{1}{2B} \text{len}(\gamma) + |\gamma^{-1}(E)|.$$

¹¹The constant 2 is only used to simplify arguments below. Any fixed bound could be used. Also, by adjusting the right-hand side by scaling with the Lipschitz constant of f , the inequality could be made homogeneous.

Here, $\gamma: [0, L] \rightarrow X$ is a 1-Lipschitz curve such that $\gamma(0) = x, \gamma(L) = z$. It is easy to see that ρ is 2-Lipschitz, and has upper gradient $g = 1_E + \frac{1}{B}$. Further $\rho(0) = 0$. By assumption, there is no path γ connecting x to y with $|\gamma| \leq Br$ and $|\gamma^{-1}|(E) \leq \frac{r}{2}$, thus either of these inequalities fails, and we have

$$\rho(y) \geq \frac{r}{2}.$$

By the Lipschitz-condition, we have

$$\rho(z) \leq \frac{r}{10},$$

for $z \in B(x, \frac{r}{20})$ and

$$\rho(z) \geq \frac{2r}{5}$$

for $z \in B(y, \frac{r}{20})$. Thus, by a doubling estimate $\frac{1}{2r} \int_{B(x, 2r)} |f - f_{B(x, 2r)}| d\mu \geq \frac{1}{20D^5}$. On the other hand

$$\begin{aligned} \int_{B(x, 2Cr)} \Phi(g) d\mu &\leq \Phi\left(\frac{1}{B}\right) + \Phi(1) \frac{\mu(E)}{\mu(B(x, Cr))} \\ &\leq \Phi\left(\sigma\left(\xi\left(\frac{1}{20D^5}\right)/2\right)\right) + D^{\log_2(B/C)+1} \Phi(1)\epsilon. \\ &\leq \xi\left(\frac{1}{20D^5}\right)/2 + D^{\log_2(B/C)+1} \Phi(1)\epsilon. \end{aligned}$$

Combining we would get

$$\xi\left(\frac{1}{20D^5}\right) \leq \xi\left(\frac{1}{20D^5}\right)/2 + D^{\log_2(B/C)+1} \Phi(1)\epsilon.$$

If $\epsilon < \frac{\xi\left(\frac{1}{20D^5}\right)}{2\Phi(1)D^{\log_2(B/C)+1}}$, this inequality fails deriving a contradiction and thus proving the conclusion. \square

Theorem 1.2. *A (D, r_0) -doubling complete metric measure space (X, d, μ) admits a local $(1, p)$ -Poincaré-inequality for some $1 \leq p < \infty$ if and only if it is locally (C, δ, ϵ) -connected for some $0 < \delta, \epsilon < 1$. Both directions of the theorem are quantitative in the respective parameters.*

Proof: The converse statement was proved in 3.16. Thus, we only prove that a measure-doubling space admitting a $(1, p)$ Poincaré-inequality for some $1 \leq p < \infty$ also admits (C, δ, ϵ) -connectivity for some constants. This is a particular case of the more general Theorem 1.4 by setting $\Phi(x) = x^p$ and $\Psi(x) = C_{PI}x^{\frac{1}{p}}$ here C_{PI} is the constant in the Poincaré inequality. However, there is an issue related to whether the Poincaré inequality is assumed for Lipschitz functions or all continuous functions and their upper gradients. We use the latter for non-homogeneous Poincaré inequalities. In the case of a complete metric space these are equivalent by [43, Theorem 2].

□

4. COROLLARIES AND APPLICATIONS

If X is a complete and proper metric space, we denote by $\Gamma(X)$ the space of geodesics parametrized by the interval $[0, 1]$. This space can be made into a complete proper metric space by using the distance $d(\gamma, \gamma') = \sup_{t \in [0, 1]} d(\gamma(t), \gamma'(t))$. Further define for each $t \in [0, 1]$ the map $e_t: \Gamma(X) \rightarrow X$ by $e_t(\gamma) = \gamma(t)$.

Definition 4.1. Let ρ be a decreasing function. A proper geodesic metric measure space (X, d, μ) is called a measure $\rho(t)$ -contraction space, if for every $x \in X$ and every $B(y, r) \subset X$, there is a Borel probability measure Π on $\Gamma(X)$ such that $e_0(\gamma) = x$, Π -almost surely, and $e_1^* \Pi = \frac{\mu|_{B(y, r)}}{\mu(B(y, r))}$, and for any $t \in (0, 1)$ we have

$$e_t^* \Pi \leq \frac{\rho(t)}{\mu(B(y, r))} d\mu.$$

This class of metric measure spaces includes the MCP-spaces by Ohta [52], as well as $CD(K, N)$ spaces, and any Ricci-limit, or Ricci-bounded space. In fact, this class of spaces is more general than the MCP-spaces considered by Ohta. Until now, it was unknown if such spaces possess Poincaré inequalities. We show as a corollary of Theorems 1.2 and 1.12 that this indeed holds. The definition could be modified to permit a probability measure on C -quasiconvex paths, but we don't need this result here.

Corollary 4.2. *Any measure $\rho(t)$ -contraction space (X, d, μ) admits a $(1, p)$ -Poincaré inequality for some $p > 0$.*

Proof: Clearly any such space is $\rho(1/2)$ -doubling. We will show that the space is $\left(2, \delta, \frac{\delta}{100\rho(1/2)^6\rho(\delta/3)}\right)$ -connected. Consider any $x, y \in X$, and $d(x, y) = r$, and a set $E \subset B(x, 2r)$ with $\mu(E) \leq \frac{\delta}{100\rho(1/2)^6\rho(\delta/3)}\mu(B(x, 2r))$. It is sufficient to find a 1-Lipschitz curve $\gamma: [0, L] \rightarrow X$ connecting x to y with $|\gamma^{-1}(E)| < \delta d(x, y)$, since then a curve fragment can be defined by restricting γ to an appropriate compact set $K \subset [0, L] \setminus E$.

Consider a midpoint z on a geodesic connecting x to y and the ball $B(z, r/2)$. Apply the definition of ρ -contraction space to a point x and the ball $B(z, r/2)$. This constructs a probability measure Π on curves from x to $B(z, r/2)$, such that

$$e_t^* \Pi \leq \frac{\rho(t)}{\mu(B(z, r/2))} d\mu.$$

Note that $\gamma(\delta/3) \in B(x, \delta/3r)$ Π -almost surely. Thus

$$\begin{aligned} \int |[\delta/3, 1] \cap \gamma^{-1}(E)| d\Pi_\gamma &= \int_{\delta/3}^1 1_E(\gamma(t)) dt d\Pi_\gamma \leq \frac{\rho(\delta/3)2}{\mu(B(z, \frac{r}{2}))} \mu(E) \\ &\leq \frac{\delta}{50}. \end{aligned}$$

Thus, with probability strictly bigger than $\frac{1}{2}$ we have $|[\delta/3, 1] \cap \gamma^{-1}(E)| \leq \frac{\delta}{10}$. Let S_x be the set of end points in $B(z, r/2)$ with such a curve to x . Similarly, construct S_y can be constructed. By the volume estimates $S_x \cap S_y \neq \emptyset$. We can

thus find a common point $w \in B(z, r/2)$, and curve fragments from $B(x, \delta/3)$ to w and from w to $B(y, \delta/3)$. This curve fragment once dilated by r results in the desired 1-Lipschitz curve fragment. \square

Corollary 4.3. *If $\rho(t) = t^{-n}$, and (X, d, μ) is a ρ -contraction space, then it satisfies a Poincaré inequality for $p > \frac{1}{n+1}$. Since MCP(0, n)-spaces satisfy this assumption we conclude that they admit such a Poincaré inequality. The same holds for every MCP(K, n)-space, but with a local Poincaré inequality.*

Proof: A slightly more detailed version of the previous proof verifies the desired connectivity estimate, and gives that the space is finely α -connected for $\alpha = \frac{1}{n+1}$. The result follows from Theorem 1.12. For $K > 0$ the latter result is obvious and $K < 0$ we modify it slightly to obtain that the space is locally finely (α, r_0) -connected (see [52] for the modified ρ). \square

Remark: It would seem that this result should be true for $p > n$, and probably $p \geq 1$, but our methods do not yield these sharper results.

We next prove a result on the existence of Poincaré inequalities on spaces deformed by Muckenhoupt-weights (recall Definition 1.16). In a geodesic doubling space any generalized $A_\infty(\mu)$ -measure is doubling. The assumptions can be slightly weakened but counter-examples can be constructed if (X, d, μ) is just doubling.

Lemma 4.4. *Let (X, d, μ) be a D -measure doubling geodesic metric measure space. Then any Radon measure $\nu \in A_\infty(\mu)$ is doubling.*

Proof: Let the parameters corresponding to ν be δ, ϵ . By [25, Lemma 3.3]¹² for $\eta \leq 2^{\frac{\log_2(1-\epsilon)}{\log_2(1+D^{-5})} - 2}$.

$$\mu(B(x, r) \setminus B(x, (1-\eta)r)) < (1-\epsilon)\mu(B(x, r)).$$

Thus $\mu(B(x, (1-\eta)r)) > \epsilon\mu(B(x, r))$, and from the Muckenhoupt condition 1.16, we get $\nu(B(x, (1-\eta)r)) > \delta\nu(B(x, r))$. Iterating this estimate $N = -1/\log_2(1-\eta) + 1$ times, we get

$$\nu(B(x, r/2)) \geq \delta^N \nu(B(x, r)).$$

This gives the desired doubling property. \square

We need another technical Lemma from the paper of Kansanen and Korte [42]. This is a metric space generalization of the classical property of self-improvement for A_∞ -weights in Euclidean space [64].

Lemma 4.5. [42] *Let (X, d, μ) be a D -measure doubling geodesic metric measure space. Then for any Radon measure $\nu \in A_\infty(\mu)$ and for any $0 < \tau < 1$ there exists a $0 < \delta < 1$, such that for any $B(x, s)$ and any $E \subset B(x, s)$*

¹²Proof works for geodesic metric spaces. See also [18, Proposition 6.12].

$$\nu(E) \leq \delta\nu(B(x, s)) \implies \mu(E) \leq \tau\mu(B(x, r)).$$

Theorem 4.6. *If (X, d, μ) is a geodesic PI-space and if $\nu \in A_\infty(\mu)$, then (X, d, ν) is also a PI-space*

Proof: By Lemma 4.5 we see that (X, d, ν) is doubling, and from Theorem 1.2 we get that (X, d, μ) is $(C, \frac{1}{2}, \epsilon_\mu)$ -connected for some C, ϵ_μ . Further by Lemma 4.5 we have a ϵ_ν such that for any Borel $E \subset B(x, r) \subset X$

$$\nu(E) \leq \epsilon_\nu\nu(B(x, r)) \implies \mu(E) \leq \epsilon_\mu\mu(B(x, r)).$$

Thus, since any obstacle E with volume density ϵ_ν with respect to ν will be an obstacle with volume density ϵ_μ with respect to μ , it is easy to verify that (X, d, ν) is $(C, \delta, \epsilon_\nu)$ -connected. All the parameters can be made quantitative. \square

5. PI-RECTIFIABILITY, ASYMPTOTIC CONNECTIVITY AND DIFFERENTIABILITY

5.1. Main theorems. In order to state our main theorem, we need to define the relevant notions. We are interested in understanding when a given metric measure space is PI-rectifiable.

Definition 5.1. A metric measure space (X, d, μ) is PI-rectifiable if there is a decomposition into sets $U_i, N \subset X$ such that

$$X = \bigcup_i U_i \cup N$$

where $\mu(N) = 0$ and there exist isometric and measure preserving embeddings $\iota_i: U_i \rightarrow \bar{U}_i$ with $(\bar{U}_i, \bar{d}_i, \bar{\mu}_i)$ PI-spaces (with possibly very different constants).

We define a differentiability space as in [18, 44].

Definition 5.2. A metric measure space (X, d, μ) is called a differentiability space (of analytic dimension $\leq M$) if there exist $U_i \subset X$ and Lipschitz functions $\phi_i: U_i \rightarrow \mathbb{R}^{n_i}$ (with $n_i \leq M$) such that $\mu(N) = 0$,

$$X = \bigcup_i U_i \cup N,$$

and such that for any Lipschitz function $f: X \rightarrow \mathbb{R}$, for every i and μ -almost every $x \in U_i$, there exists a *unique* linear map $D^{\phi_i} f(x): \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ such that

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)).$$

A stronger definition is obtained by assuming differentiability of Lipschitz functions with certain infinite dimensional targets [8].

Definition 5.3. A metric measure space (X, d, μ) is a RNP-differentiability space (of analytic dimension $\leq N$) if there exist $U_i \subset X$ and associated Lipschitz functions $\phi_i: U_i \rightarrow \mathbb{R}^{n_i}$ (with $n_i \leq N$) such that $\mu(N) = 0$,

$$X = \bigcup_i U_i \cup N,$$

and such that for any Banach space V with the Radon-Nikodym property and any Lipschitz function $f: X \rightarrow V$, for every i and μ -almost every $x \in U_i$ there exists a *unique* linear map $D^{\phi_i} f(x): \mathbb{R}^{n_i} \rightarrow V$ such that

$$f(y) = f(x) + D^{\phi_i} f(x)(\phi_i(y) - \phi_i(x)) + o(d(x, y)).$$

Definition 5.4. A Banach space V has the Radon-Nikodym property if every Lipschitz function $f: [0, 1] \rightarrow V$ is almost everywhere differentiable.

For several equivalent and sufficient conditions, consult for example [53] or the references in [21].

Definition 5.5. A metric measure space (X, d, μ) is called asymptotically well-connected if for all $0 < \delta < 1$ and almost every point x there exists $r_x > 0$, $0 < \epsilon_x < 1$ and $C_x \geq 1$ such that for any $y \in X$ such that $d(x, y) < r_x$ the pair (x, y) is $(C_x, \delta, \epsilon_x)$ -connected.

We can now state a general theorem that characterizes PI-rectifiable metric measure spaces. This is proven later in this section.

Theorem 5.6. *A complete metric measure space (X, d, μ) is a PI-rectifiable if and only if it is asymptotically doubling and asymptotically well-connected.*

Using this we can prove the PI-rectifiability result.

Theorem 1.1. *A complete metric measure space (X, d, μ) is a RNP-Lipschitz differentiability space if and only if it is PI-rectifiable and every porous set has measure zero.*

Proof of Theorem 1.1: By [8, Theorem 3.8] we have (up to a change in terminology and some refinements in their arguments) that a RNP-Lipschitz differentiability space is asymptotically doubling and asymptotically well-connected. For the converse, we refer to the main result in [22], where it is shown that a PI space is a RNP-Lipschitz differentiability space. It is then trivial to conclude that a PI-rectifiable space is also a RNP-Lipschitz differentiability space if every porous set has measure zero. See the discussion at the end of the introduction in [7] as well as the arguments in [9].

□

The proof of Theorem 5.6 is based on a general “thickening Lemma”. To see this, we outline the proof. By using measure theory arguments, the asymptotic connectedness can be used to produce subsets such that the space X satisfies some doubling and connectivity estimates uniformly along such sets. Then these subsets are enlarged, or “thickened” to improve a relative form of connectivity to an intrinsic form of connectivity. This step is included in the following theorem.

Theorem 1.18. *Let $r_0 > 0$ be arbitrary. Assume (X, d, μ) is a metric measure space and $K \subset A \subset X$, A is measurable and K compact. Assume further that X is (D, r_0) -doubling along A , A is uniformly $(\frac{1}{2}, r_0)$ -dense in X along K , and A with the restricted measure and distance is locally $(C, 2^{-60}, \epsilon, r_0)$ -connected¹³ along K . There exists constants $\bar{C}, \bar{\epsilon}, \bar{D} > 0$, and a complete metric space \bar{K} which*

¹³By an iteration argument similar to Theorem 1.11, or as presented in [8], the constant 2^{-60} -could be replaced by any $0 < \delta < 1$, but this would make our proof more technical and is unnecessary.

is \overline{D} -doubling and $(\overline{C}, \frac{1}{2}, \overline{\epsilon}, r_0 2^{-20})$ -connected, and an isometry $\iota: K \rightarrow \overline{K}$ which preserves the measure. In particular, the resulting metric measure space \overline{K} is a PI-space.

The enlarged space \overline{K} is obtained by attaching a tree-like metric space T to it. We remark, that there is no unique “thickening” constructing, but it is our belief that gluing a tree-like metric space is the easiest way to obtain the desired conclusions. Additionally, the previous Lemma, when applied to subsets $K \subset X$ of PI-spaces X , produces new examples of PI-spaces from old-ones, where the possible disconnectedness of K is repaired by the glued space T .

Our primary goal is constructing T is increasing connectivity and making \overline{K} quasiconvex. That goal could be attained by attaching an edge for every pair of points in K . This, however, would fail to be doubling. Thus, when need more care in constructing T . Our construction is focused on the following issues.

- (1) The resulting space needs to be a complete locally compact metric measure space. In particular, the added intervals need measures associated with them.
- (2) The measures of the added intervals need to be controlled by μ in order to preserve doubling.
- (3) Doubling needs to be controlled. We shouldn't add too many intervals at any given scale and location.
- (4) The intervals should themselves be “simple” PI-spaces, i.e. the measures should be related to Lebesgue measure.
- (5) A curve fragment in A between points in K should be replaceable, up to small measure, by a possibly somewhat longer curve fragment in the glued space.

A natural construction to obtain these goals arises from a modification of a so called hyperbolic filling. This construction first appeared in [17], and later in [14].

We will first indicate how Theorem 5.6 follows from Theorem 1.18.

Proof of Theorem 5.6: By general measure theory arguments, we can construct sets $X = \bigcup_i V_i \cup N = \bigcup_{i,j} K_i^j \cup N'$, such that each V_i are measurable, $K_i^j \subset V_i$ are compact, $\mu(N) = \mu(N') = 0$ and the following uniform estimates hold.

- X is (D_i, r_i) -doubling along V_i .
- X is $(C_i, 2^{-60}, \epsilon_i, r_i)$ -connected along V_i for some $\epsilon_i < 1$.
- V_i is $(1 - \epsilon_i/2, r_i^j)$ -dense in X along K_i^j .

For the definitions of these concepts see the beginning of Section 2.

Since the set V_i is uniformly $(1 - \epsilon_i/2, r_i^j)$ -dense in X along K_i^j , directly we obtain that V_i with its restricted measure and metric is $(C_i, 2^{-60}, \epsilon_i/2, \min(r_i, r_i^j)/C_i)$ -connected along K_i^j . To see this, verify definition 1.8 for every obstacle E by adjoining the complement $X \setminus V_i$ to any set E considered. By the density bound we see that $E \cup X \setminus V_i$ satisfies a slightly worse density bound, and thus is a permissible obstacle.

Theorem 1.18 can now be applied to $A = V_i$, $K = K_i^j$ and $\epsilon = \epsilon_i/2, r_0 = \min(r_i, r_i^j)/C_i = r_0^{i,j}, C = C_i$ and $D = D_i$ to obtain isometric and measure preserving embeddings $\iota: K_i^j \rightarrow \overline{K}_i^j$ to metric measure spaces $(\overline{K}_i^j, d_i^j, \mu_i^j)$. Further, by the same theorem, there exist positive constants $\overline{D}_{i,j}, \overline{C}_{i,j}$ and $\overline{\epsilon}_{i,j}$ such that the metric measure spaces $(\overline{K}_i^j, d_i^j, \mu_i^j)$ are locally $(\overline{D}_{i,j}, r_0^{i,j} 2^{-20})$ -doubling and locally

$(\bar{C}_{i,j}, 2^{-1}, \bar{\epsilon}_{i,j}, r_0^{i,j} 2^{-20})$ -connected. Thus, by the main Theorem 3.16 each \bar{K}_i^j is a PI-space. This completes the proof of PI-rectifiability. \square

5.2. Construction. Continue next with the proof of Theorem 1.18. First, we present the construction, which is followed by a separate subsection containing the proofs of the relative connectivity and doubling properties.

“Thickening” construction for Theorem 1.18: Assume for simplicity that $D \geq 2$. Re-scale so that $r_0 = 2^{20}$. We will define $\bar{K} = K \cup T$, where T arises as the metric space induced by a metric graph (V, E) , and K naturally sits at the boundary of T . The vertices will arise from two types of points, net points of K and gap points that correspond to center points in $A \setminus K$.

First define the 2^{-j} -nets N_j of K for $j \geq 0$, i.e. maximal collections of points such that $d(n, m) \geq 2^{-j}$ for $n, m \in N$. In order to make some notation below consistent define $N_{-1} = \emptyset$. Define these inductively such that $N_j \subset N_{j+1}$, and define $N = \bigcup_j N_j$ the collection of all net points. Further, define the scale function as $\mathbf{Sc}_N(n) = 2^{-i}$ if $n \in N_i \setminus N_{i-1}$

Using Vitali covering Lemma [31,64], choose “gap” points $g_k^m \in (N_1(K) \setminus K) \cap A$, indexed by $(k, m) \in U \subset \mathbb{N}_0 \times \mathbb{N}_0$, such that

$$(5.7) \quad (N_1(K) \setminus K) \cap A \subset \bigcup_{(k,m) \in U} B(g_k^m, 2^{-k-10}),$$

$$(5.8) \quad 2^{-k-1} < d(g_k^m, K) \leq 2^{-k}$$

and for $(k, m) \neq (k', m')$ with $(k, m), (k', m') \in U$

$$(5.9) \quad B(g_k^m, 2^{-k-15}) \cap B(g_{k'}^{m'}, 2^{-k'-15}) = \emptyset.$$

Recall that $N_1(K) = \{x \in X \mid d(x, K) < 1\}$. This type of covering is also known as a Whitney-covering. Given such a covering, define the gap points at scale 2^{-k} by $G_k = \{g_k^m \mid (k, m) \in U\}$ and $G = \bigcup_k G_k$ the collection of all gap points. We will define a scale function by $\mathbf{Sc}_G(g) = 2^{-k}$ for $g \in G_k$.

Define the vertex sets as $V = V_G \cup V_N$, where V_G arises from G and V_N arises from N . Vertices will be defined as pairs (x, r) with $x \in X$ and $r > 0$. First take

$$V_G = \{(g, 2^{-k}) \mid g \in G_k\}.$$

These will also be referred to as bridge-points because they are used to repair the gap corresponding to the center g at scale 2^{-k} .

Next, define for each $n \in N$ the maximal scale at which there is a near-by gap point.

$$(5.10) \quad l(n) = \max \{2^{-k} \mid \exists g \in G_k : d(g, n) \leq 2^{4-k} \text{ and } \mathbf{Sc}_G(g) = 2^{-k} \leq \mathbf{Sc}_N(n)\}.$$

If there is no such point, we will define $l(n) = 0$. We define the vertex set arising from the net points as

$$V_N = \{(n, 2^{-k}) | n \in N, 2^{-k} \leq l(n)\}.$$

Define the edge set E as follows. Two distinct vertices $(x, r), (y, s) \in V$ are connected¹⁴, i.e. $e = ((x, r), (y, s)) \in E$ if

$$(5.11) \quad d(x, y) \leq 2^4(r + s),$$

and

$$(5.12) \quad \frac{1}{2} \leq r/s \leq 2.$$

We define a simplicial complex T with edges corresponding to $e \in E$ and vertices corresponding to $v \in V$. In this geometric realization use I_e to represent the interval corresponding to e . For each $v \in V$ we represent the corresponding point in T with the same symbol. The simplicial complex becomes a metric space by declaring the length of the edge $e = ((a, s), (b, t))$ to be

$$(5.13) \quad |e| = 2^4(s + t).$$

A metric d_T is induced by the path metric on the simplicial complex. Since T may not be connected, some pairs of points $v, w \in T$ may have $d_T(v, w) = \infty$.

When a point x lies on an interval I_e , we will often abuse notation and say $x \in e$. Also, we will use the word edge to refer either to the symbol e associated to it, or the interval I_e in the geometric realization. Define the set which will be our metric space as $\bar{K} = K \cup T$. We define a symmetric function δ defined on a subset of $\bar{K} \times \bar{K}$ and the metric \bar{d} on \bar{K} to be the maximal metric subject to

- $\forall x, y \in K : \bar{d}(x, y) \leq d(x, y) = \delta(x, y)$
- $\forall n \in N, (n, s) \in V_N : \bar{d}(n, (n, s)) \leq 3 \cdot 2^4 s = \delta(n, (n, s))$
- $\forall v, w \in T : \bar{d}(v, w) \leq d_T(v, w) = \delta(v, w)$

The distance can be given explicitly by a minimization over discrete paths. Let $x, y \in \bar{K}$ and $\sigma = (\sigma_0, \dots, \sigma_m)$ be a sequence of points in \bar{K} . The variable m is called the length of the path. We call such a sequence a discrete path in \bar{K} , and say that it connects x to y , if $\sigma_0 = x, \sigma_m = y$. We call it admissible, if for each consecutive pair σ_i, σ_{i+1} , either they both lie in T or K , or one of them is equal to n , for some $n \in N$, and the other is equal to (n, s) for $(n, s) \in V_N$. In other words, we require $\delta(\sigma_i, \sigma_{i+1})$ to be defined by one of the cases above. Denote by $\Sigma_{x,y}^n$ the space of all admissible discrete paths of length n that connect x to y . With these definitions, the distance becomes

$$(5.14) \quad \bar{d}(x, y) = \inf_n \inf_{\sigma \in \Sigma_{x,y}^n} \sum_{i=0}^{n-1} \delta(\sigma_i, \sigma_{i+1}).$$

A measure on each interval $I_e = [0, |e|]$, which is associated to the edge $e = ((x, r), (y, s))$, is defined as weighted Lebesgue measure:

¹⁴A minor technical point is that we use tuples. Thus, really the resulting space is a directed graph. We could also use undirected graphs, but it will simplify notation below to allow both edges.

$$(5.15) \quad \mu_e = \frac{\mu(B(x, r)) + \mu(B(y, s))}{|e|} \lambda.$$

The total measure is $\mu_T = \sum_e \mu_e$, where each measure μ_e is extended to be zero on the complement of I_e , and the measure on the new space \bar{K} is defined as μ_T on T and $\mu|_K$ on K . In other words

$$\bar{\mu} = \mu_T + \mu|_K.$$

5.3. Proof that $(\bar{K}, \bar{d}, \bar{\mu})$ is a PI-space. Our goal is to show that the space constructed in the previous subsection satisfies the properties from Theorem 1.18. This involves some preliminary lemmas.

Definition 5.16. Let $L > 1$. A metric space (X, d) is said to be uniformly (L, r_0) -perfect if for any $B(x, r)$ with $0 < r < r_0$ the following holds

$$B(x, r_0) \setminus B(x, r) \neq \emptyset \implies B(x, r) \setminus B(x, r/L) \neq \emptyset.$$

The space X is said to be uniformly (L, r_0) -perfect along a subset $S \subset X$ if the same holds for any $x \in S$ and $0 < r < r_0$.

Lemma 5.17. *Let $r_0, C, \epsilon > 0$ be arbitrary. If A is a metric measure space and A is $(C, \delta, \epsilon, r_0)$ -connected along $K \subset A$ for some $0 < \delta < \frac{1}{8}$, then A is uniformly $(7/5, r_0)$ -perfect along K .*

Proof: Choose an arbitrary $x \in K$ and $0 < r < r_0$. Assume $B(x, r_0) \setminus B(x, r) \neq \emptyset$. Let $s = \inf_{y \in A \setminus B(x, 5r/7)} d(x, y)$. By assumption $s < r_0$. If $s < r$, we are done. For the sake of contradiction, assume that $s \geq r$. Choose y such that $d(x, y) < 8s/7$ and $d(x, y) < r_0$. By the definition of s we know that $B(x, s) \setminus B(x, 6s/7)$ is empty. Then, by the connectivity assumption (and choosing $E = \emptyset$), there is a 1-Lipschitz curve fragment $\gamma: K' \rightarrow A$ connecting x and y with $|\text{Undef}(\gamma)| < \frac{1}{8}d(x, y)$. Assume $a = \sup_{t \in K', \gamma(t) \in B(x, 6s/7)} t$ and $b = \inf_{a < t \in K', \gamma(t) \in A \setminus B(x, s)}$. Then

$$d(x, y)/8 < s/7 \leq d(\gamma(a), \gamma(b)) \leq |b - a|.$$

Further, the interval $(a, b) \subset \mathbb{R} \setminus K'$. Namely, if $t \in K' \cap (a, b)$, we have $\gamma(t) \in B(x, s)$ (by definition of b), but then automatically $\gamma(t) \in B(x, 6s/7)$, which contradicts the definition of a . Then we conclude that $|b - a| \leq |\text{Undef}(\gamma)| < d(x, y)/8$, and this is a contradiction. □

This lemma is mainly used to give upper bounds for volumes in a doubling metric measure space.

Lemma 5.18. *Let $r_0, D > 0$ be arbitrary. Assume that X is a complete metric space, X is (D, r_0) -doubling along $A \subset X$ and A is uniformly $(7/5, r_0)$ -perfect along a set $K \subset A$. Then if $x \in K$, $0 < r < r_0$ and $A \cap B(x, r_0) \setminus B(x, r) \neq \emptyset$,*

$$(5.19) \quad \mu(B(x, r/2)) \leq (1 - \sigma)\mu(B(x, r))$$

for $\sigma = \frac{1}{D^4}$.

Proof: By the previous lemma (applied for $7r/8$) and the assumption there is a point $y \in A \cap B(x, 7r/8) \setminus B(x, 5r/8)$. For such a y we have $B(y, r/8) \subset B(x, r) \setminus B(x, r/2)$. Also, by doubling

$$\mu(B(y, r/8)) \geq \frac{\mu(B(x, r))}{D^4},$$

which gives

$$\mu(B(x, r/2)) \leq \mu(B(x, r)) - \mu(B(y, r/8)) \leq \left(1 - \frac{1}{D^4}\right) \mu(B(x, r)).$$

□

We will also need a lemma concerning concatenating curve fragments.

Lemma 5.20. *Assume (X, d, μ) is a (D, r_0) -doubling metric space. If $1 > \epsilon, \delta > 0$, $L, C \geq 1$ are fixed and $p_i \in X$, for $i = 1, \dots, n$, are points such that each pair (p_i, p_{i+1}) is (C, δ, ϵ) -connected and $\sum_{i=1}^{n-1} d(p_i, p_{i+1}) \leq Ld(p_1, p_n)$, then (p_1, p_n) is*

$$(LC, 2L\delta, \epsilon D^{\log_2(\delta) - \log_2(L) - \log_2(n) - \log_2(C) - 6})\text{-connected.}$$

Proof: Denote $r = d(p_1, p_n)$, and assume $E \subset B(p_1, LCr)$ with

$$\mu(E \cap B(p_1, LCr)) \leq \epsilon D^{\log_2(\delta) - \log_2(L) - \log_2(n) - \log_2(C) - 6} \mu(B(p_1, LCr)).$$

Define $l(1) = 0$ and $l(j) = \sum_{i=1}^{j-1} d(p_i, p_{i+1})$ for $j = 2, \dots, n$. Also, define the set of intervals $\mathcal{I} = \{(Cl(i), Cl(i+1)) \mid i = 1, \dots, n-1\}$ and $\mathcal{G} = \{I \in \mathcal{I} \mid |I| \geq \frac{\delta r}{n}\}$. For each $I = (Cl(i), Cl(i+1)) \in \mathcal{G}$ we have $d_i = d(p_i, p_{i+1}) \geq \frac{\delta r}{nC}$. In particular, by doubling

$$\mu(E \cap B(p_i, Cd_i)) \leq \epsilon D^{\log_2(\delta) - \log_2(L) - \log_2(n) - \log_2(C) - 6} \mu(B(p_1, LCr)) \leq \epsilon \mu(p_i, Cd_i).$$

Thus, we can define $\gamma_I: K_I \rightarrow X$ to be a 1-Lipschitz curve fragment connecting p_i to p_{i+1} with $K_I \subset I$, $\text{len}(\gamma_I) \leq |I| = Cd_i$, $|\text{Undef}(\gamma_I)| < \delta d_i$ and $\gamma_I(I) \cap E = \emptyset$. Further, assume by a slight dilation and translation that $\min(K_I) = \min(I)$, $\max(K_I) = \max(I)$. Define $K = \{l(j)\} \cup \bigcup_{I \in \mathcal{G}} K_I$. Next, define a 1-Lipschitz curve fragment $\gamma: K \rightarrow X$ by setting $\gamma(l(i)) = p_i$ and $\gamma(t) = \gamma_I(t)$ for $t \in I \in \mathcal{G}$. Clearly $\text{len}(\gamma) \leq \max(K) \leq LCr$, and $\text{Undef}(\gamma) \subset \bigcup_{I \in \mathcal{I} \setminus \mathcal{G}} I \cup \bigcup_{I \in \mathcal{G}} \text{Undef}(\gamma_I)$. Thus,

$$|\text{Undef}(\gamma)| \leq n \frac{\delta d(x, y)}{n} + \delta \sum_{i=1}^{n-1} d_i < 2L\delta r.$$

Since p_i might belong to E , we might need to remove small neighborhoods of these points to satisfy the third condition in 1.8. This modification of γ is trivial.

□

Further, since our space \overline{K} arises by gluing a tree to K , the following lemma is useful.

Lemma 5.21. *Assume (X, d, μ) is a (D, r_0) -doubling metric space, and $I \subset X$ is isometric to a bounded interval in \mathbb{R} . Assume further that the restricted measure is given as $\mu|_I = c\lambda$ where λ is the induced Lebesgue measure on I and $c > 0$ is some constant, and that for any sub-interval $[a, b] \in I$ we have $B((a+b)/2, (b-a)/2) \subset I$. Then for any $\delta > 0$ and any $x, y \in I$ we have that (x, y) is $(1, \delta, \delta(2D)^{-2})$ -connected.*

Proof: Let $x, y \in I$. Denote by $r = d(x, y)$ and by z the midpoint of x and y on the interval I . Take an arbitrary Borel set E with

$$\frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} \leq \delta/(2D)^2$$

We will connect the pair of points x, y by the geodesic segment $\gamma: J \rightarrow X$ where $J \subset I$ is the sub-interval defined by x and y . Note that $J = B(z, r/2)$. The curve γ may intersect with E , but

$$\begin{aligned} |\gamma^{-1}(E)| &\leq \lambda(J \cap E) \leq \frac{1}{c} \mu(J \cap E) \\ &\leq \lambda(J) \frac{\mu(J \cap E)}{\mu(J)} \leq r \frac{\mu(E \cap B(x, r))}{\mu(B(z, r/2))} \\ &\leq rD^2 \frac{\mu(E \cap B(x, r))}{\mu(B(x, r))} \leq rD^2 \delta / (2D)^2 = \delta r/2. \end{aligned}$$

On the third line we used the fact that $J = B(z, r/2)$.

Thus, by restricting γ to an almost full measure sub-set in the complement of the set $\gamma^{-1}(E)$, we obtain the desired curve fragment. □

Occasionally, we will need to vary the constant C in Definition 1.8. Thus, we use the following.

Lemma 5.22. *Assume (X, d, μ) is a (D, r_0) -doubling metric space and $0 < \delta, \epsilon, C \geq 1$ and $K \geq 0$ given constants. If $(x, y) \in X \times X$ with $d(x, y) \leq r_0 2^{-K}/C$ is (C, δ, ϵ) -connected then it is also $(2^K C, \delta, \epsilon D^{-K-1})$ -connected.*

Proof: Denote by $r = d(x, y)$. Take an arbitrary Borel set E with

$$\frac{\mu(B(x, 2^K Cr) \cap E)}{\mu(B(x, 2^K Cr))} \leq \epsilon D^{-K-1}.$$

Then doubling gives also

$$\frac{\mu(B(x, Cr) \cap E)}{\mu(B(x, Cr))} \leq \epsilon.$$

Thus, the result follows from the assumption and Definition 1.8. □

Proof that the construction in Subsection 5.2 works for Theorem 1.18:

Preliminaries: Continue using the notation of Subsection 5.2. For future reference, we compute some basic estimates and define some related notation. Define

the scale function $\mathbf{Sc}: V \rightarrow \mathbb{R}$ which is given by $\mathbf{Sc}((x, r)) = r$ and location function $\mathbf{Loc}: V \cup K \rightarrow K$ which assigns for $(x, r) \in V$ the value $\mathbf{Loc}((x, r)) = x$ and restricts to identity on K . For a vertex $v = (x, r) \in V$ we denote the set $E_v = \{e \in E \mid e \text{ is incident to } v\}$.

Lemma 5.23. *For all $v \in V$ we have*

$$(5.24) \quad |E_v| \leq 4D^{25}.$$

To see this consider $V_v^N = \{w \mid w \in V_N, (v, w) \in E\}$ and $V_v^G = \{w \mid w \in V_G, (v, w) \in E\}$. Then $|E_v| \leq 2|V_v^N| + 2|V_v^G|$. For $w \in V_v^G$, then $\mathbf{Sc}(w) \geq r/2$, and $d(\mathbf{Loc}(w), \mathbf{Loc}(v)) \leq 2^6 r$. In particular, by the disjointedness property in Equation (5.9), the points $\mathbf{Loc}(w)$ have pairwise distance at least a $r2^{-15}$ in $B(\mathbf{Loc}(v), 2^6 r) \cap A$. Thus, their number is bounded by D^{25} , i.e. $|V_v^G| \leq D^{25}$.

Similarly, if $w \in V_v^N$ we have $\mathbf{Sc}(w) \geq r/2$. In particular, $l(\mathbf{Loc}(w)) \geq r/2$. Thus, the pairwise distances of distinct $\mathbf{Loc}(w)$ are at least $r/2$. Also,

$$d(\mathbf{Loc}(w), \mathbf{Loc}(v)) \leq 2^6 r.$$

The number of such w is bounded by $|V_v^N| \leq D^9$. We get by summing up the estimates for V_v^G and V_v^N that

$$|E_v| \leq 4D^{25}.$$

Lemma 5.25. *The following estimates are true on \bar{K} .*

- (1) For every $x, y \in K \subset \bar{K}$: $\bar{d}(x, y) = d(x, y)$.
- (2) For every $e \in E$ and every $x, y \in I_e \subset T$ we have $\bar{d}(x, y) = d_T(x, y)$.
- (3) For every $(x, r) \in V$ there is a $n \in N$ such that $e = ((n, r), (x, r)) \in E$. In particular, $\bar{d}((x, r), n) \leq 2^6 r$.
- (4) If $(x, r) \in V$ and $y \in K$, then $\max\{r, d(x, y)\} \leq \bar{d}((x, r), y) \leq d(x, y) + 2^8 r$.
- (5) If $(x, r), (y, s) \in V$ are distinct, then $\max\{d(x, y), r + s\} \leq \bar{d}((x, r), (y, s)) \leq d(x, y) + 2^4(r + s)$.
- (6) If $x \in e = (v, w) \in E$ and $y \in \bar{K} \setminus e$, then $\bar{d}(x, y) = \min\{\bar{d}(x, v) + \bar{d}(v, y), \bar{d}(x, w) + \bar{d}(w, y)\}$.
- (7) If $v \in V$, and $e \in E_v$, then $|e| \leq 2^7 \mathbf{Sc}(v)$ and $\mu(B(\mathbf{Loc}(v), \mathbf{Sc}(v))) \leq \bar{\mu}(I_e) \leq 2D^7 \mu(B(\mathbf{Loc}(v), \mathbf{Sc}(v)))$.
- (8) For any $n \in N$ and any $0 < r \leq l(n)$ we have

$$\sum_{2^l \leq r} \sum_{e \in E_{(n, 2^l)}} \bar{\mu}(I_e) \leq 2^{20} D^{37} \mu(B(n, r)).$$

- (9) There is a function $\rho: N \rightarrow G \cup N$ such that $\rho(n) = n$ if $l(n) = 0$, and if $l(n) > 0$, then $\rho(n) \in G_{l(n)}$ and

$$d(\rho(n), n) \leq 2^4 l(n).$$

Further, for any $g \in G$ we have $\rho^{-1}(g) \leq D^{25}$.

Proof: We will proceed in numerical order.

Estimate 1: Take arbitrary $x, y \in K$. It is obvious that $\bar{d}(x, y) \leq d(x, y)$. For an arbitrary $\epsilon > 0$ we can find a $n \in \mathbb{N}$ and $\sigma \in \Sigma_{x, y}^n$ such that

$$\bar{d}(x, y) + \epsilon \geq \sum_i \delta(\sigma_i, \sigma_{i+1}).$$

We can assume by possibly making the sum on the right smaller that for all $i = 0, \dots, n$ it holds that $\sigma_i \in K$ or $\sigma_i \in V \subset T$. Thus, there is a function $f: \{0, \dots, n\} \rightarrow K$, given by $f(i) = \sigma_i$ if $\sigma_i \in K$ and $f(i) = \mathbf{Loc}(\sigma_i)$ if $\sigma_i \in V$. By the definition of δ preceding (5.14) we have for $i = 0, \dots, n-1$

$$d(f(i), f(i+1)) \leq \delta(\sigma_i, \sigma_{i+1}).$$

By the triangle inequality

$$\bar{d}(x, y) + \epsilon \geq \sum_i d(f(i), f(i+1)) \geq d(x, y).$$

Since $\epsilon > 0$ is arbitrary $\bar{d}(x, y) \geq d(x, y)$, which shows $\bar{d}(x, y) = d(x, y)$.

Estimate 2: The proof, that we have for any points $x, y \in T$ on a common edge $e \in E$ the equality $d_T(x, y) = \bar{d}(x, y)$, is a trivial application of the definition of δ and observing that a discrete path that does not directly connect the points x, y in e will traverse an entire edge adjacent to both end points of e , or two separate edges adjacent to each end point of e . The length of such a path is larger than $|e| \geq d_T(x, y)$.

Estimate 3: Next, take an arbitrary $(x, r) \in V$. We wish to find a near-by $n \in N$. Either $x \in N$ or $x \in G$. In the first case define $x = n$, and by the definition of \bar{d} we have

$$\bar{d}(n, (x, r)) \leq 3 \cdot 2^4 r \leq 2^6 r.$$

Assume that $x = g \in G$. Then by (5.8) there is a $k \in K$ such that $d(x, k) \leq r$. Let $n \in N$ be a net point closest to k with $r \leq 2\mathbf{Sc}(n)$. Then $d(n, k) \leq r$. Thus, by the triangle inequality $d(g, n) \leq 2r \leq 2^2\mathbf{Sc}(n)$, and by (5.10) we have $l(n) \geq r$. Then both (n, r) and (g, r) are vertices in V joined by an edge. Further, $\bar{d}((n, r), (g, r)) = 2^5 r$ by Estimate 2 above and (5.13). By the triangle inequality and what we just observed $\bar{d}(n, (g, r)) \leq 2^6 r$.

Estimate 4: Take a $y \in K$ and $(x, r) \in V$. Choose $n \in N$ such that $\bar{d}((x, r), n) \leq 2^6 r$. We have $d(y, n) \leq d(x, n) + d(y, x) \leq 2^6 r + d(y, x)$. Thus, $\bar{d}((x, r), y) \leq 2^7 r + d(x, y)$.

In order to obtain the desired lower bound, consider the height function given by $h: \bar{K} \rightarrow \mathbb{R}$ and defined as $h(x) = 0$ for $x \in K$ and by $h(v) = r$ for $v = (x, r) \in V$. On each edge extend f linearly. Now for any $x, y \in \bar{K}$ for which δ was defined (see discussion preceding (5.14)) we get $|h(x) - h(y)| \leq \delta(x, y)$. Thus, for any discrete path σ connecting x to y we obtain

$$|h(x) - h(y)| \leq \sum_i \delta(\sigma_i, \sigma_{i+1}).$$

Taking an infimum over σ we get $|h(x) - h(y)| \leq \bar{d}(x, y)$. A particular case of this gives when $(x, r) \in V$ and $y \in K$

$$r = |h((x, r)) - h(y)| \leq \bar{d}((x, r), y).$$

The lower bound $d(x, y) \leq \bar{d}((x, r), y)$ is proven similarly, to the corresponding one in Estimate 5.

Estimate 5: Let $(x, r), (y, s) \in V$ be distinct. Using Estimate 3 we can find $n_x, n_y \in K$ such that $\bar{d}((x, r), n_x) \leq 2^6 r$ and $\bar{d}((y, s), n_y) \leq 2^6 s$. Further, by the calculations for Estimate 3 we get $d(n_x, x) \leq 2r$ and $d(n_y, y) \leq 2r$. A trivial application of the triangle inequality then gives $\bar{d}((x, r), (y, s)) \leq 2^8 r + d(x, y)$. Next consider the lower bound. Let σ be any discrete path that passes through vertices in V and points in K . The discrete path must traverse an edge adjacent to (x, r) , and one adjacent to (y, s) . These edges might be the same, or different, but in either case one can verify that their total length is at least $r + s$, and thus $r + s \leq \bar{d}((x, r), (y, s))$. Next, with the triangle inequality we obtain

$$d(x, y) \leq \sum_i d(\mathbf{Loc}(\sigma_i), \mathbf{Loc}(\sigma_{i+1})) \leq \sum_{i=0}^{n-1} \delta(\sigma_i, \sigma_{i+1}).$$

Taking the infimum gives $d(x, y) \leq \bar{d}(x, y)$, which completes the proof.

Estimate 6: Trivial from the definition in (5.14).

Estimate 7: These estimates follow directly from doubling and (5.11) and (5.12).

Estimate 8: Take an arbitrary $n \in N$ and $r \leq l(n)$. Let $r \leq L = 2^k \leq l(n)$ such that $L \leq 2r$. For any $s \leq l(n)$ and any $e \in E_{(n, s)}$ using Estimate 6 of this lemma we obtain

$$(5.26) \quad \begin{aligned} \mu_e(I_e) &= \mu(B(n, s)) + \mu(B(y, t)) \\ &\leq 2D^7 \mu(B(n, s)). \end{aligned}$$

Apply Lemma 5.23 to get

$$\begin{aligned} \sum_{2^l \leq r} \sum_{e \in E_{(n, 2^l)}} \bar{\mu}(I_e) &\leq \sum_{2^l \leq L} \sum_{e \in E_{(n, 2^l)}} \bar{\mu}(I_e) \\ &\leq \sum_{2^l \leq L} 8D^{32} \mu(B(n, 2^l)) \end{aligned}$$

The last sum can be estimated using Lemma 5.49 to get

$$\sum_{2^l \leq L} 8D^{32} \mu(B(n, 2^l)) \leq 2^{20} D^{36} \mu(B(n, L)) \leq 2^{20} D^{37} \mu(B(n, r)).$$

The factor 2^{10} arises from the fact that the non-emptiness assumption of Lemma 5.49 can be guaranteed only for $s \leq L2^{-4} \leq l(n)2^{-4}$.

Estimate 9: In the case $l(n) = 0$, define $\rho(n) = n$. By the definition of $l(n)$ in (5.10), for each $n \in N$ such that $l(n) > 0$ we can find a $g \in G_{l(n)}$ such that $d(g, n) \leq 2^4 l(n)$. In this case, define $\rho(n) = g$ by choosing one of them. This defines a function $\rho: N \rightarrow G \cup N$. Next, if $g \in G$ we can use the definition of $l(n)$ to obtain two properties. Firstly, for any $n \in \rho^{-1}(g)$ we have the estimate $d(g, n) \leq 2^4 \mathbf{Sc}(g)$. Secondly, if $n, m \in \rho^{-1}(g)$ are distinct then $d(n, m) \geq \mathbf{Sc}(g) 2^{-15}$ (by (5.9)). Thus, by doubling their total number can be bounded by doubling as $|\rho^{-1}(g)| \leq D^{25}$. \square

Next, we show completeness by taking an arbitrary Cauchy sequence x_i and finding a limit point. If infinitely many of the x_i lie in K , or a single edge I_e , the result follows trivially. Thus, we can assume by passing to a sub-sequence that each $x_i \in e_i = ((a_i, r_i), (b_i, s_i))$ for distinct edges $e_i \in E$. We can assume $r_i \rightarrow 0$, since by the compactness of K we obtain that for any given $\epsilon > 0$ there are only finitely many edges with $r_i > \epsilon$. By estimate (5.8) we have $d(a_i, K) \leq r_i$, and thus by Estimate 4 in 5.25 there is a $k \in K$ such that

$$\bar{d}((a_i, r_i), k_i) \leq 2^9 r_i.$$

On the other hand by (5.11) and the second estimate in Lemma 5.25 we get

$$\bar{d}((a_i, r_i), x_i) \leq 2^9 r_i + \bar{d}((a_i, r_i), (b_i, s_i)) \leq 2^9 r_i + d(a_i, b_i) + 2^8 (r_i + s_i) \leq 2^{15} r_i.$$

In particular, $\bar{d}(x_i, k_i) \leq 2^{20} r_i$. Thus, we can conclude that k_i is also a Cauchy sequence in K and has a limit point k_∞ . It is now easy to see that $\lim_{i \rightarrow \infty} x_i = k_\infty$.

Our desired metric measure space is $(\bar{K}, \bar{d}, \bar{\mu})$. The remainder of the proof shows that it is a PI-space. We have already observed that \bar{K} is complete. Since K and T are totally bounded with respect to \bar{d} it is not hard to see that \bar{K} is also totally bounded and thus compact. The measure is a sum of finite Radon measures and thus as long as it's bounded it will itself be a Radon measure. This boundedness follows from estimates below. It remains to show the doubling and connectivity properties. The last step is to apply Theorem 3.16 to conclude that the resulting space is a PI-space.

In order to distinguish between metric and measure notions on \bar{K} and those on the original space X we will often use a line above the symbol. For example for $x \in K$, the metric ball in X is denoted $B(x, r)$, and the ball in \bar{K} is denoted $\bar{B}(x, r)$. There is no risk of confusing this with the closure of the ball because we will not be using that in any of the arguments below.

Doubling: The doubling condition 2.2 depends on two parameters (x, r) and we need to check three cases depending on the location and scale. This is done by estimating the volume of various balls from above and below.

Case 1: $x \in K$ and $0 < r < \frac{1}{2}$:

- Lower bound for $x \in K, 0 < r < 1$: We will use two disjoint subsets of $\bar{B}(x, r)$: B_K (part in K) and B_T (corresponding to a portion of the glued intervals). In precise terms we set

$$\overline{B}_K = \overline{B}(x, r) \cap K$$

and

$$\overline{B}_T = \bigcup_{\substack{v \in V_G, e \in E_v \\ \overline{d}(v, x) < r2^{-10}}} I_e.$$

If $y \in \overline{B}_T$, then there is a $v \in V$ such that $y \in e \in E_v$ and $\overline{d}(v, x) < r2^{-10}$. By Cases 3, 6 and 7 of Lemma 5.25 we have $\overline{d}(x, y) \leq |e| + \overline{d}(x, v) \leq r2^{-3} + r2^{-10} < r$. Thus

$$(5.27) \quad \overline{B}_K \cup \overline{B}_T \subset \overline{B}(x, r).$$

The sets \overline{B}_K and \overline{B}_T are disjoint, and we can estimate the total volume from below by estimating each of them individually and summing the estimates.

$$(5.28) \quad \overline{\mu}(\overline{B}_K) = \mu(B(x, r) \cap K)$$

The volume of \overline{B}_T can be estimated using Case 7 and Case 4 of Lemma 5.25.

$$(5.29) \quad \overline{\mu}(\overline{B}_T) \geq \bigcup_{\substack{v=(g,s) \in V_G, e \in E_v \\ \overline{d}(v,x) < r2^{-10}}} \mu_e(I_e)$$

$$(5.30) \quad \geq \sum_{\substack{v=(g,s) \in V_G, e \in E_v \\ d(g,x) < r2^{-20}}} \mu(B(g, s))$$

$$(5.31) \quad \geq \mu(B(x, r2^{-26}) \cap A \setminus K).$$

In the last line we used the covering property

$$B(x, r2^{-26}) \cap A \setminus K \subset \bigcup_{\substack{g \in G \\ d(x,g) \leq r2^{-20}}} B(g, \mathbf{Sc}(g)2^{-10}).$$

To see this take an arbitrary $z \in B(x, r2^{-13}) \cap A \setminus K$. Then by the Whitney covering property (5.7) there is a $g \in G$ such that

$$z \in B(g, \mathbf{Sc}(g)2^{-10}).$$

However, $d(g, x) \geq \mathbf{Sc}(g)2^{-1}$ by (5.8) and $d(g, x) \leq r2^{-26} + \mathbf{Sc}(g)2^{-10}$ by the triangle inequality. Thus, we get $\mathbf{Sc}(g) \leq r2^{-25}$ and $d(g, x) \leq r2^{-25}$. This is sufficient to verify the previous inclusion.

We obtain using estimates (5.28) and (5.31) that

$$(5.32) \quad \begin{aligned} \overline{\mu}(\overline{B}(x, r)) &\geq \overline{\mu}(B_K) + \overline{\mu}(B_T) \\ &\geq \mu(B(x, r2^{-26}) \cap A \setminus K) + \mu(B(x, r2^{-26}) \cap A \cap K) \\ &\geq \mu(B(x, r2^{-26}) \cap A) \geq \frac{1}{D^{264}} \mu(B(x, r)). \end{aligned}$$

The uniform density of A for $x \in K$ was used on the last line.

- Upper bound for $x \in K, 0 < r < 1$: Define the following sets of edges:

$$E_2 = \{e \in E \mid \exists v = (n, s) \in V_N, e \in E_v, d(n, x) \leq r, s \leq l(n) \leq r\},$$

$$E_3 = \{e \in E \mid \exists v = (n, s) \in V_N, e \in E_v, d(n, x) \leq r, l(n) \geq r \geq s\},$$

$$E_4 = \{e \in E \mid \exists v = (g, s) \in V_G, e \in E_v, d(g, x) \leq r\}.$$

By the definition of the space and simple distance estimates from Cases 1–7 in Lemma 5.25, we can decompose the set into pieces as follows.

$$\overline{B}(x, r) \subset A_1 \cup A_2 \cup A_3 \cup A_4,$$

with $A_1 = \overline{B}(x, r) \cap K$,

$$A_2 = \bigcup_{e \in E_2} I_e,$$

$$A_3 = \bigcup_{e \in E_3} I_e$$

and

$$A_4 = \bigcup_{e \in E_3} I_e.$$

First we get the trivial upper bound

$$(5.33) \quad \bar{\mu}(A_1) = \mu(B(x, r) \cap K) \leq \mu(B(x, r)).$$

Use Case 8 of Lemma 5.25 to obtain

$$(5.34) \quad \begin{aligned} \bar{\mu}(A_2) &= \sum_{e \in E_2} \bar{\mu}(I_e) \\ &\leq \sum_{\substack{n \in N \\ d(n, x) \leq r \\ 0 < l(n) \leq r}} \sum_{\substack{s < l(n) \\ e \in E_{(n, s)}}} \bar{\mu}(I_e) \\ &\leq \sum_{\substack{n \in N \\ d(n, x) \leq r \\ 0 < l(n) \leq r}} 2^{12} D^{22} \mu(B(n, l(n))) \end{aligned}$$

Next, we use the map from Case 9 of Lemma 5.25 and doubling to bound $\mu(B(n, l(n))) \leq D^6 \mu(B(\rho(n), \mathbf{S}\mathbf{c}_G(\rho(n))2^{-4}))$, and $d(x, \rho(n)) \leq 2^5 r$. Also, the balls $B(\rho(n), \mathbf{S}\mathbf{c}_G(\rho(n))2^{-1.5}) \subset B(x, 2^5 r)$ are disjoint by (5.9). All in all, we can apply a union bound with (5.34) to conclude

$$\begin{aligned}
\bar{\mu}(A_2) &\leq \sum_{\substack{n \in N \\ d(n,x) \leq r \\ 0 < l(n) \leq r}} 2^{12} D^{22} \mu(B(n, l(n))) \\
&\leq 2^{12} D^{40} \sum_{\substack{n \in N \\ d(n,x) \leq r \\ 0 < l(n) \leq r}} \mu(B(\rho(n), \mathbf{Sc}_G(\rho(n)) 2^{-15})) \\
&\leq 2^{12} D^{40} \sum_{\substack{g \in G \\ d(g,x) \leq 2^5 r}} \sum_{n \in \rho^{-1}(g)} \mu(B(\rho(n), \mathbf{Sc}_G(\rho(n)) 2^{-4})) \\
&\leq 2^{12} D^{60} \sum_{\substack{g \in G \\ d(g,x) \leq 2^5 r}} \mu(B(g, \mathbf{Sc}_G(g) 2^{-4})) \\
(5.35) \quad &\leq 2^{12} D^{60} \mu(B(x, 2^5 r)) \leq 2^{12} D^{65} \mu(B(x, r))
\end{aligned}$$

Next, we estimate $\bar{\mu}(A_3)$. Use Case 8 of Lemma 5.25 to obtain

$$\begin{aligned}
\bar{\mu}(A_3) &= \sum_{e \in E_3} \bar{\mu}(I_e) \\
&\leq \sum_{\substack{n \in N \\ d(n,x) \leq r \\ r \leq l(n)}} \sum_{\substack{s \leq r \\ e \in \bar{E}(n,s)}} \bar{\mu}(I_e) \\
(5.36) \quad &\leq 2^{12} D^{23} \sum_{\substack{n \in N \\ d(n,x) \leq r \\ r \leq \mathbf{Sc}_N(n)}} \mu(B(n, r/2))
\end{aligned}$$

Since $\mathbf{Sc}_N(n) \geq r$, the net-points n included in the sum are r -separated. Therefore, the balls $B(n, r/2)$ are disjoint. Also, $B(n, r/2) \subset B(x, 2r)$. We obtain

$$\begin{aligned}
\bar{\mu}(A_3) &\leq 2^{12} D^{23} \sum_{\substack{n \in N \\ d(n,x) \leq r \\ r \leq \mathbf{Sc}_N(n)}} \mu(B(n, r/2)) \\
(5.37) \quad &\leq 2^{12} D^{23} \mu(B(x, 2r)) \leq 2^{12} D^{24} \mu(B(x, r))
\end{aligned}$$

Finally, we estimate $\bar{\mu}(A_4)$. Consider some $v = (g, r) \in V_G$ and $e \in E_v$ such that $d(g, x) \leq r$. By (5.8) we have $\mathbf{Sc}(g)/2 \leq r$. Then by Case 7 $\bar{\mu}(I_e) \leq 2D^7 \mu(B(g, \mathbf{Sc}(g))) \leq 2D^{25} \mu(B(g, \mathbf{Sc}(g) 2^{-15}))$. For different v the balls $B(g, \mathbf{Sc}(g) 2^{-15})$ are disjoint by (5.9) and $B(g, \mathbf{Sc}(g) 2^{-15}) \subset B(x, 4r)$. Combine all these to see that

$$\begin{aligned}
\bar{\mu}(A_4) &= \sum_{e \in E_3} \bar{\mu}(I_e) \\
(5.38) \quad &\leq 2D^{25} \mu(B(x, 4r)) \leq 2D^{30} \mu(B(x, r))
\end{aligned}$$

Combining the estimates (5.33), (5.35), (5.37), (5.38), with doubling for the different sets gives

$$(5.39) \quad \bar{\mu}(\bar{B}(x, r)) \leq 2^{16} D^{65} \mu(B(x, r)).$$

- Combine estimates (5.32) and (5.39) for $x \in K, 0 < r < 1/2$ to give

$$(5.40) \quad \frac{\bar{\mu}(\bar{B}(x, 2r))}{\bar{\mu}(\bar{B}(x, r))} \leq 2^{20} D^{100}.$$

Case 2: $x \in e_x = ((a, s), (b, t)) \in E, 0 < r < s/4$: Denote $v = (a, s)$ and $w = (b, t)$. Let E_v, E_w be the sets of edges adjacent to either vertex. It is easy to conclude that

$$(5.41) \quad \bar{B}(x, r) \subset \bar{B}(x, 2r) \subset \bigcup_{e \in E_v \cup E_w} I_e.$$

For each $e \in E_v \cup E_w$ and any $h > 0$ we have $\text{diam}(B(x, h) \cap e) \leq 2h$. Also, $|e| \geq s$ by (5.13). Further, we derive from (5.15) that in terms of densities

$$\mu_e \leq \frac{2\mu(B(a, 2^{10}s))}{|e|} \lambda \leq \frac{2\mu(B(a, 2^{10}s))}{s} \lambda.$$

Thus using $h = 2r$, the size bound Lemma 5.23 we get

$$(5.42) \quad \begin{aligned} \bar{\mu}(\bar{B}(x, 2r)) &= \sum_{e \in E_v \cup E_w} \mu(I_e \cap B(x, 2r)) \\ &\leq \frac{8r}{s} \mu(B(a, 2^{10}s)) \sum_{e \in E_v \cup E_w} 1 \\ &\leq \frac{2^{10} D^{35} r \mu(B(a, s))}{s}. \end{aligned}$$

On the other hand for e_x we have $\text{diam}(B(x, r) \cap e_x) \geq r$, and using $|e| \leq 2^7 s$ from Case 7 in Lemma 5.25 we see that in terms of densities

$$\mu_e \geq \frac{\mu(B(a, s))}{|e|} \lambda \geq \frac{\mu(B(a, s))}{2^7 s} \lambda.$$

Using this we obtain the lower bound

$$(5.43) \quad \begin{aligned} \bar{\mu}(\bar{B}(x, r)) &\geq \mu_e(e \cap B(x, r)) \\ &\geq \frac{r \mu(B(a, s))}{2^7 s}. \end{aligned}$$

Combine estimates (5.43) and (5.42) to get the desired doubling bound

$$\frac{\bar{\mu}(\bar{B}(x, 2r))}{\bar{\mu}(\bar{B}(x, r))} \leq 2^{20} D^{35}.$$

Case 3: $x \in e = ((a, s), (b, t)) \in E, s/4 < r < 2^{12}s$ and $r < 2^{-13}$: Let $v = (a, s)$. By the estimate (5.43) and using that for $r2^{-12} < s/4$, we get

$$(5.44) \quad \bar{\mu}(\bar{B}(x, r)) \geq \bar{\mu}(\bar{B}(x, r2^{-14})) \geq \frac{r\mu(B(a, s))}{2^{30}r} \geq 2^{-30}\mu(B(a, r2^{-12})).$$

By Case 3 of Lemma 5.25 there is a $n \in N$ such that $\bar{d}(v, n) \leq 2^6 s$. In particular, $\bar{d}(n, x) \leq \bar{d}(v, n) + |e| \leq 2^7 s + 2^6 s \leq 2^{10} s$. Apply the estimate (5.39) to see

$$(5.45) \quad \bar{\mu}(\bar{B}(x, 2r)) \leq \bar{\mu}(\bar{B}(n, 2^{11}r)) \leq 2^{15}D^{65}\mu(B(n, 2^{11}r)).$$

Using doubling with Cases 3 and 4 in Lemma 5.25 we get $\mu(B(n, 2^{11}r)) \leq \mu(B(a, 2^{12}r)) \leq D^{25}\mu(B(a, 2^{-12}r))$. Finally, combining this with (5.44) we get the desired doubling bound

$$\frac{\bar{\mu}(\bar{B}(x, 2r))}{\bar{\mu}(\bar{B}(x, r))} \leq 2^{45}D^{90}.$$

Case 4: $x \in e = ((a, s), (b, t)) \in E$, $2^{12}s < r$ and $r < 1/8$: Using Cases 3 and 7 of Lemma 5.25 we can find a n

$$\bar{d}(x, n) \leq 2^8 s \leq r2^{-4}.$$

In particular, we have

$$\bar{B}(n, r2^{-4}) \subset \bar{B}(x, r) \subset \bar{B}(x, 2r) \subset \bar{B}(n, 4r).$$

Thus using (5.40), we obtain

$$\frac{\bar{B}(x, 2r)}{\bar{B}(x, r)} \leq \frac{\bar{\mu}(\bar{B}(n, 4r))}{\bar{\mu}(\bar{B}(n, r2^{-4}))} \leq 2^{200}D^{500}.$$

These cases conclude the proof for doubling. Denote the maximum of the previous doubling constants $\bar{D} = 2^{200}D^{500}$. The cases above show that $(\bar{K}, \bar{d}, \bar{\mu})$ is $(\bar{D}, 2^{-13})$ -doubling.

Connectivity: Next, we move to prove connectivity. We will show that every pair of points (x, y) , which is sufficiently close, is $(\bar{C}, \frac{1}{2}, \bar{\epsilon})$ -connected for appropriately chosen $\bar{\epsilon}$ and \bar{C} . Again we have cases depending on the positions of x and y . The most complicated case is when both $x, y \in K$.

We always denote $r = \bar{d}(x, y)$. For each pair of points take an arbitrary open set E such that $\bar{\mu}(E \cap \bar{B}(x, Cr)) \leq \eta\bar{\mu}(\bar{B}(x, Cr))$, where η, C are the relevant connectivity parameters in each case. Also, it is enough to verify Definition 1.8 for open sets, because of the regularity of the measure. Here, we have already shown that $\bar{\mu}$ is a bounded measure on a compact metric space, and thus $\bar{\mu}$ is a Radon measure, and both inner and outer regular (see [51]). Note, by doubling and Lemma 5.22, we don't need to be so concerned about the parameter C , which represents the length of the curve fragments, being the same in each case. At the end \bar{C} will become the maximum from the different C 's in each case, and we will adjust $\bar{\epsilon}$ to account for this.

For technical reasons, some cases involve seemingly stronger statements than necessary. Again, we will be somewhat liberal in our estimates. For example, we

will use \bar{D} in places where D would suffice. Together these cases show that each $(x, y) \in \bar{X} \times \bar{X}$ with $\bar{d}(x, y) \leq 2^{-200}C^{-1}$ is

$$\left(2^{35}C, 2^{-14}, \epsilon(2\bar{D})^{-2000-2\log_2(C)}\right)\text{-connected.}$$

Case 1: Let $\delta > 0$ be arbitrary and assume $(x, y) \in T \times T$ with $\bar{d}(x, y) < 2^{-6}$, and $x \in e = ((a, s), (b, t)) \in E$. If $\bar{d}(x, y) < s2^{-4}$, then the pair (x, y) is $(1, \delta, \delta(2\bar{D})^{-10+\log_2(\delta)})$ -connected.

Let $v = (a, s)$ and $w = (b, t)$. By Case 2 in 5.25 and (5.13) we know that $x, y \in \cup_{e \in E_v \cup E_w} I_e = A$. Lemma 5.23 gives that A is a connected subgraph of T with at most $4D^{13}$ edges. The metric restricted to A agrees with the path metric d_T by similar arguments to those used in Estimate 2 in Lemma 5.25. The path connecting any pair of points passes through at most three intervals. The result follows now easily from Lemmas 5.21 and 5.20.

Case 2: Every pair $(x, y) \in K \times K$ with $d(x, y) \leq 2^{-200}C^{-1}$ is

$$\left(2^{20}C, 2^{-30}, \epsilon(2\bar{D})^{-1000-\log_2(C)}\right)\text{-connected.}$$

Let $E \subset \bar{B}(x, 2^{20}Cd(x, y))$ be a set such that

$$(5.46) \quad \bar{\mu}(E) \leq \epsilon(2\bar{D})^{-1000-\log_2(C)} \bar{\mu}(\bar{B}(x, 2^{20}Cd(x, y))).$$

Our goal will be to first construct a curve fragment in A and then to replace portions of it with a curve in T . This latter step is only possible, if the portions of the curve fragment in A doesn't pass too close to certain "bad" gap points. First, we define bad bridge points.

Consider the collection \mathbf{B} of "bad" bridge points $b \in V$ with $\mathbf{Sc}(b) < 2^{-50}$ such that

$$(5.47) \quad \bar{\mu}(\bar{B}(b, 2^{10}\mathbf{Sc}(b)) \cap E) \geq (2\bar{D})^{-\log_2(C)-500} \bar{\mu}(\bar{B}(b, 2^{10}\mathbf{Sc}(b))).$$

Define

$$\mathcal{B} = \bigcup_{b \in \mathbf{B}} B(\mathbf{Loc}(b), \mathbf{Sc}(b))$$

and its approximant in \bar{K}

$$\bar{\mathcal{B}} = \bigcup_{b \in \mathbf{B}, e \in E_b} I_e.$$

We will seek to estimate the volume of \mathcal{B} from above. Each of the edges I_e can appear at most twice in the union defining $\bar{\mathcal{B}}$. Use Case 7 of Lemma 5.25 to see that

$$\begin{aligned}
\mu(\mathcal{B}) &\leq \sum_{b \in \mathbf{B}} \mu(B(\mathbf{Loc}(b), \mathbf{Sc}(b))) \\
&\leq \sum_{b \in \mathbf{B}, e \in E_b} \bar{\mu}(I_e) \\
(5.48) \quad &\leq 2\bar{\mu}(\bar{\mathcal{B}}).
\end{aligned}$$

We can use the Vitali covering Lemma [31, 64] to choose a sub-collection \mathbf{B}' of $b \in \mathbf{B}$ such that

$$\bar{\mathcal{B}} \subset \bigcup_{b \in \mathbf{B}'} \bar{B}(b, 2^{13}\mathbf{Sc}(b))$$

and the balls $\bar{B}(b, 2^{10}\mathbf{Sc}(b))$ for $b \in \mathbf{B}'$ are disjoint. Apply doubling of $\bar{\mu}$, which we have already found, and the definition (5.47) to get the following.

$$\begin{aligned}
\bar{\mu}(E) &\geq \sum_{b \in \mathbf{B}'} \bar{\mu}(\bar{B}(b, 2^{10}\mathbf{Sc}(b)) \cap E) \\
&\geq (2\bar{D})^{-\log_2(C)-500} \sum_{b \in \mathbf{B}'} \bar{\mu}(\bar{B}(b, 2^{10}\mathbf{Sc}(b))) \\
&\geq (2\bar{D})^{-\log_2(C)-503} \sum_{(k,m) \in \mathbf{B}'} \bar{\mu}(\bar{B}(b, 2^{13}\mathbf{Sc}(b))) \\
(5.49) \quad &\geq (2\bar{D})^{-\log_2(C)-503} \bar{\mu}(\bar{\mathcal{B}})
\end{aligned}$$

Thus, using (5.46) with doubling we see

$$(5.50) \quad \bar{\mu}(\bar{\mathcal{B}}) \leq \epsilon (2\bar{D})^{-400} \bar{\mu}(\bar{B}(x, Cd(x, y))).$$

Thus with the estimates (5.46), (5.48) and (5.33), we see that

$$(5.51) \quad \mu(\mathcal{B} \cup (K \cap E)) \leq 2\bar{\mu}(\bar{\mathcal{B}}) + \bar{\mu}(E) \leq \epsilon \mu(B(x, Cd(x, y))).$$

By using the connectivity condition, we can find a 1-Lipschitz curve fragment $\gamma: S \rightarrow A$, which is parametrized by length (see Lemma 2.15) and which satisfies $\gamma(0) = x, \gamma(\text{len}(\gamma)) = y, |\text{Undef}(\gamma)| < 2^{-60}d(x, y), \text{len}(\gamma) \leq Cd(x, y)$ and $\gamma^{-1}(\mathcal{B} \cup (K \cap E)) \subset \{0, \text{max}(S)\}$.

The image of the curve fragment γ may not be contained in K . Thus, to reach our desired conclusion we will define another curve fragment $\bar{\gamma}$ in \bar{K} by replacing the portions in A/K with curves in \bar{K} . This is done in two steps. First we discretize the portions of the fragment in $A \setminus K$ and obtain a “discrete path fragment” through gap points g . Intuitively, the sub-segments of the curve fragment associated to gap points g will be related to the corresponding bridge point $b \in V$. In the construction, we can guarantee that $b \notin \mathbf{B}$, and thus we will be able to connect sufficiently many consecutive bridge points to each other and go from a discrete path to a continuous curve fragment. This latter part of the argument is called extension. Finally, the curve fragment we construct won’t satisfy the desired Lipschitz-bound so we will need to dilate the domain by a certain factor and thus complete the proof.

- (1) **Discretization:** First, we cover the set $\gamma^{-1}(A \setminus K) = O$ by intervals as follows. Recall the Whitney covering property (5.7), the length estimate, and the fact that $\gamma(O) \cap \mathcal{B} = \emptyset$. Using these, choose for each $z \in O$ a g_z with the property $d(g_z, \gamma(z)) \leq \mathbf{Sc}_G(g_z)2^{-10}$. It is easy to see that $b_z = (g_z, \mathbf{Sc}_G(g_z)) \notin \mathbf{B}$. Define the interval

$$(5.52) \quad I_z = (z - \mathbf{Sc}_G(g_z)2^{-10}, z + \mathbf{Sc}_G(g_z)2^{-10})$$

and the smaller interval $J_z = (z - \mathbf{Sc}_G(g_z)2^{-15}, z + \mathbf{Sc}_G(g_z)2^{-15})$ and choose discrete centers $\bar{Z} \subset O$ such that

$$O \subset \bigcup_{z \in \bar{Z}} I_z,$$

and J_z are pairwise disjoint for $z \in \bar{Z}$. This is possible by the Vitali covering theorem [31, 64]. Because γ is 1-Lipschitz, we obtain $I_z \subset [0, \max(S)] \setminus \gamma^{-1}(K)$ for all $z \in \bar{Z}$.

We will define a “discrete curve fragment” as follows. Consider the compact set $\bar{S}_0 = \gamma^{-1}(K) \cup \bar{Z}$ and the curve fragment $\bar{\gamma}_0: \bar{S}_0 \rightarrow \bar{K}$ by setting $\bar{\gamma}_0(t) = \gamma(t)$ for $t \in S$ and $\bar{\gamma}_0(z) = b_z$ for $z \in \bar{Z}$. The crucial properties we need are:

- (a) $\max(\bar{S}_0) = \max(\bar{S}) \leq Cd(x, y)$,
- (b) $\text{LIP}(\bar{\gamma}_0) \leq 2^{20}$.

The first estimate is immediate from the definition. Since the curve fragment is 1-Lipschitz on $\gamma^{-1}(K)$ (this involves Estimate 1 from Lemma 5.25), it is sufficient to check the Lipschitz-bound for pairs of points $z, w \in \bar{Z}$ and $z \in \bar{Z}$ and $w \in \gamma^{-1}(K)$.

Consider the case where both $z, w \in \bar{Z}$. Then by the triangle inequality

$$d(g_z, g_w) \leq d(g_z, \gamma(z)) + d(\gamma(z), \gamma(w)) + d(g_w, \gamma(w)).$$

Further by the Lipschitz bound $d(\gamma(z), \gamma(w)) \leq |z - w|$, and by the choice of g_z and g_w , for $t = z, w$ we get $d(g_t, \gamma(t)) \leq \mathbf{Sc}_G(g_t)2^{-10}$. Also, $J_z \cap J_w = \emptyset$, so $\mathbf{Sc}_G(g_z)2^{-15} + \mathbf{Sc}_G(g_w)2^{-15} \leq |z - w|$. All these combined give $d(g_z, g_w) \leq 2^7|z - w|$.

Use Estimate 6 from Lemma 5.25 to give the Lipschitz-bound.

$$\begin{aligned} \bar{d}(\bar{\gamma}_0(z), \bar{\gamma}_0(w)) &= \bar{d}(b_z, b_w) \\ &\leq d(g_z, g_w) + 2^4(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)) \\ &\leq 2^{20}|z - w|. \end{aligned}$$

Consider next the case $z \in \bar{Z}$ and $w \in \gamma^{-1}(K)$. Then by the triangle inequality

$$d(g_z, \gamma(w)) \leq d(g_z, \gamma(z)) + d(\gamma(z), \gamma(w)).$$

Further by the Lipschitz bound $d(\gamma(z), \gamma(w)) \leq |z - w|$, and by the choice of g_z and Estimate (5.8) we get $d(g_z, \gamma(z)) \leq \mathbf{Sc}_G(g_z)2^{-10} \leq d(g_z, K)2^{-9} \leq d(g_z, \gamma(w))2^{-9}$. Thus

$$d(g_z, \gamma(w)) \leq 2|z - w|.$$

Also, $\mathbf{Sc}_G(g_z) \leq 2d(g_z, K) \leq 4|z - w|$. Use Estimate 4 from Lemma 5.25 for $b_z = (g_z, \mathbf{Sc}_G(g_z))$ to give the Lipschitz-bound

$$\bar{d}(\bar{\gamma}_0(z), \bar{\gamma}_0(w)) = \bar{d}(b_z, \gamma(w)) \leq d(g_z, \gamma(w)) + 2^8 \mathbf{Sc}_G(g_z) \leq 2^{20}|z - w|.$$

- (2) **Extension:** The desired curve fragment $\bar{\gamma}$ will result from expanding $\bar{\gamma}_0$ for certain “consecutive” pairs of $z, w \in \bar{Z}$. We call $(z, w) \in \bar{S}_0 \times \bar{S}_0$ *consecutive* if $z < w$ and $(z, w) \cap \bar{S}_0 = \emptyset$. For each consecutive pair $(z, w) \in \bar{Z} \times \bar{Z}$ consider the interval $I_{z,w} = [z, w]$. Let $\mathcal{I} = \{I_{z,w}\}$ be the collection of all such intervals. Then, let \mathcal{G} be the collection of “good” intervals $I_{z,w}$ such that $z, w \in \bar{Z}$ and

$$(5.53) \quad d(g_z, g_w) \leq 2^{-3}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)),$$

and finally let $\mathcal{F} = \mathcal{I} \setminus \mathcal{G}$.

Define $\bar{S} = \bar{S}_0 \cup \bigcup_{I \in \mathcal{G}} I$, and the curve fragment $\bar{\gamma}: \bar{S} \rightarrow \bar{K}$ by $\bar{\gamma}(t) = \bar{\gamma}_0(t)$ for $t \in \bar{S}_0$, and $\bar{\gamma}$ is a linear parametrization of the edge $I_{(b_z, b_w)} \subset T$ for $I_{z,w} \in \mathcal{G}$. This edge exists by the following argument.

Firstly, $\mathbf{Sc}(g_z)/2 < d(g_z, K) \leq d(g_z, g_w) + d(g_w, K)$. Thus, $\mathbf{Sc}(g_z)3/8 < \frac{9}{8}\mathbf{Sc}(g_w)$ follows from estimates (5.53) and (5.8). By symmetry, we obtain $|\mathbf{Sc}(g_z)/\mathbf{Sc}(g_w)| < 4$, and thus (5.12) must hold. On the other hand, ensuring (5.11) is a triviality.

- (3) **Estimates:** We will show that this curve fragment satisfies the following.

- (a) $\text{LIP}(\bar{\gamma}) \leq 2^{20}$
- (b) $\text{len}(\bar{\gamma}) \leq 2^{20}Cd(x, y)$
- (c) $|\text{Undef}(\bar{\gamma})| < 2^{-31}d(x, y)$
- (d) $|\bar{\gamma}^{-1}(E)| < 2^{-31}d(x, y)$

From these the desired curve fragment satisfying the conditions of Definition 1.8 can be obtained by restricting $\bar{\gamma}$ to a large compact set in the complement of $\bar{\gamma}^{-1}(E)$ and by dilating the domain by a factor of 2^{20} . Thus, we are left with proving these four estimates.

The Lipschitz estimate is trivial, since the domain is expanded by linear parametrizations. Here we also use Estimate 2 in Lemma 5.25. Similarly, the length estimate follows because $\text{len}(\bar{\gamma}) = \text{len}(\bar{\gamma}_0) \leq 2^{20} \max(\bar{S}_0) \leq 2^{20} \text{len}(\gamma)$.

Next, estimate $\text{Undef}(\bar{\gamma})$. It is easy to see $\text{Undef}(\bar{\gamma}) \subset \text{Undef}(\gamma) \cup \bigcup_{I \in \mathcal{F}} I$. Let $I \in \mathcal{F}$ be an arbitrary interval. Consider the interval $C_I = (z, w) \setminus (I_z \cup I_w)$. Clearly $C_I \subset \text{Undef}(\gamma)$, because $C_I \cap K = \emptyset$. We will show in the following paragraph that $|I| \leq 2|C_I|$. Assuming this for now, we will complete the estimate. Since the intervals $I \in \mathcal{F}$ are disjoint, so are $C_I \subset I$. Also,

$$\begin{aligned}
|\text{Undef}(\bar{\gamma})| &\leq |\text{Undef}(\gamma)| + \sum_{I \in \mathcal{F}} |I| \\
&\leq |\text{Undef}(\gamma)| + \sum_{I \in \mathcal{F}} 2|C_I| \\
&\leq 3|\text{Undef}(\gamma)| \leq 2^{-31}d(x, y).
\end{aligned}$$

For each $I = I_{z,w} \in \mathcal{F}$ we have $d(g_z, g_w) > 2^{-3}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w))$ and $(z, w) \cap \bar{S} = \emptyset$. Thus, $|z - w| \geq d(\gamma(z), \gamma(w)) \geq 2^{-4}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w))$. Also, from definition (5.52) and the previous, we get

$$\begin{aligned}
|C_I| &= |(z, w) \setminus (I_z \cup I_w)| \\
&\geq 2^{-4}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)) - 2^{-10}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)) \\
&\geq 2^{-5}(\mathbf{Sc}_G(g_z) + \mathbf{Sc}_G(g_w)).
\end{aligned}$$

Finally, from (5.52),

$$|I| \leq |C_I| + 2^{-10}\mathbf{Sc}_G(g_z) + 2^{-10}\mathbf{Sc}_G(g_w) \leq 2|C_I|.$$

The remaining estimate concerns $|\bar{\gamma}^{-1}(E)|$. For each $I = (z, w) \in \mathcal{G}$ denote by $e_I = (b_z, b_w)$ the edge in $T \subset \bar{K}$ corresponding to it. Note that

$$(5.54) \quad |e_I| \leq 2^{20}|z - w| = 2^{20}|I|$$

by the Lipschitz estimate of $\bar{\gamma}$. Assume without loss of generality that $\mathbf{Sc}(b_w) \leq \mathbf{Sc}(b_z)$. Since $b_z, b_w \notin \mathcal{B}$, we have by Estimate (5.47)

$$(5.55) \quad \bar{\mu}(\bar{B}(b_z, 2^{10}\mathbf{Sc}(b_z)) \cap E) < (2\bar{D})^{-\log_2(C)-500} \bar{\mu}(\bar{B}(b_z, 2^{10}\mathbf{Sc}(b_z))).$$

Thus, by doubling estimates similar to Lemma 5.21, estimate $\mathbf{Sc}(b_w) \leq \mathbf{Sc}(b_z)$ and estimate (5.13) we have

$$(5.56) \quad \bar{\mu}(e_I \cap E) \leq (2\bar{D})^{-\log_2(C)-400} \bar{\mu}(e_I).$$

Since $\bar{\gamma}$ is a linear parametrization of e_I with the interval I , we have also

$$(5.57) \quad |\bar{\gamma}^{-1}(E) \cap I| \leq (2\bar{D})^{-\log_2(C)-300} |I|.$$

Thus, we get

$$\begin{aligned}
|\bar{\gamma}^{-1}(E)| &\leq \sum_{I \in \mathcal{G}} |\bar{\gamma}^{-1}(E) \cap I| \\
&\leq \sum_{I \in \mathcal{G}} (2\bar{D})^{-\log_2(C)-300} |I| \\
&\leq (2\bar{D})^{-\log_2(C)-300} Cd(x, y) \leq 2^{-31}d(x, y).
\end{aligned}$$

Case 3. For any $\delta > 0$ and every $x \in e = ((a, s), (b, t)) \in T$ with $s < 2^{-6}$ there is a $y \in K$ such that $\bar{d}(x, y) \leq 2^8s$ and such that (x, y) is $(2^9, \delta, \delta(2\bar{D})^{2\log_2(\delta)-40})$ -connected. Let $v = (a, s), w = (b, t)$. By Case 3 in Lemma 5.25 there is a $n \in N$ such that $\bar{d}(n, v) \leq 2^6s$. Further, $((n, s), v) \in E$. Set $y = n$.

Next, take $k = \lceil 20 - \log_2(\delta) \rceil$. Define the points $p_1 = x, p_2 = v, p_{k+1} = n = y$ and $p_i = (n, s2^{3-i})$ for $i = 3, \dots, k$. By Case 5 or Lemma 5.25 we have

$$\bar{d}(p_k, y) \leq 2^8 \mathbf{Sc}(p_k) \leq 2^{\log_2(\delta)-5} s \leq \delta/4 \bar{d}(x, y).$$

Each pair (p_i, p_{i+1}) are edges in T by condition (5.11) and (5.12). Thus, by Lemma 5.21, each pair (p_i, p_{i+1}) , for $1 \leq i \leq k-1$, is $(1, \delta 2^{-11}, \delta(2\bar{D})^{-14})$ -connected. Also,

$$\sum_{i=1}^{k-1} \bar{d}(p_i, p_{i+1}) \leq 2^8 s,$$

and $\bar{d}(p_1, p_k) \geq s/2$. Further by Lemma 5.20 (using $L = 2^9$) and distance estimates from Lemma 5.25, the pair (p_1, p_k) is $(2^9, \delta/2, \delta(2\bar{D})^{-30-\log_2(\delta)-\log_2(k)})$ -connected. We observe that $\bar{d}(p_k, p_{k+1}) \leq \delta/4 \bar{d}(x, y)$. Thus, any curve fragment connecting p_1 to p_k can be enlarged by a small gap to one connecting p_1 to p_{k+1} . This changes the length slightly, so by repeating an argument from Lemma 5.22, we see that the pair $(x, y) = (p_1, p_{k+1})$ is $(2^9, \delta, \delta(2\bar{D})^{2\log_2(\delta)-40})$ -connected.

Case 4. Each pair $(x, y) \in T \times T$ with $0 < \bar{d}(x, y) \leq 2^{-100} C^{-1}$ is

$$(2^{35} C, 2^{-14}, \epsilon(2\bar{D})^{-2000-2\log_2(C)})\text{-connected.}$$

Let $x \in e_x = (v_x, w_x)$ and $y \in e_y = (v_y, w_y)$. If $\bar{d}(x, y) < \mathbf{Sc}(v_x)2^{-4}$ or $\bar{d}(x, y) < \mathbf{Sc}(v_y)2^{-4}$, the result follows from Case 1. Thus, assume that

$$\bar{d}(x, y) \geq \max(\mathbf{Sc}(v_x)2^{-4}, \mathbf{Sc}(v_y)2^{-4}).$$

Let n_x, n_y be as in Case 3. In other words $\bar{d}(n_x, x) \leq 2^8 \mathbf{Sc}(v_x) \leq 2^{12} \bar{d}(x, y)$, $\bar{d}(n_y, y) \leq 2^8 \mathbf{Sc}(v_y) \leq 2^{12} \bar{d}(x, y)$ and (n_x, x) and (n_y, y) are $(2^9, 2^{-30}, (2\bar{D})^{-200})$ -connected. Now, consider the discrete path $p_1 = x, p_2 = n_x, p_3 = n_y, p_4 = y$. Note, that (p_2, p_3) is $(2^{20} C, 2^{-30}, \epsilon(2\bar{D})^{-1000-\log_2(C)})$ -connected by Case 2. We have

$$\sum_{i=1}^3 \bar{d}(p_i, p_{i+1}) \leq 2^{15} \bar{d}(x, y).$$

Thus from Lemma 5.20, we get that (x, y) is $(2^{35} C, 2^{-14}, \epsilon(2\bar{D})^{-2000-2\log_2(C)})$ -connected.

Case 5. For each $x \in T$ and $y \in K$ the pair (x, y) with $d(x, y) \leq 2^{-100} C^{-1}$ is

$$(2^{35} C, 2^{-14}, \epsilon(2\bar{D})^{-2000-2\log_2(C)})\text{-connected.}$$

Define n_x as before. Then we can consider the discrete path given by $p_1 = x, p_2 = n_x$ and $p_3 = y$. The result follows analogously to Case 4 from Lemma 5.20.

Concluding remarks: We have established $(\bar{D}, 2^{-5})$ -doubling and

$$(2^{35} C, 2^{-14}, \epsilon(2\bar{D})^{-2000-2\log_2(C)}, 2^{-100} C^{-1})\text{-connectivity.}$$

By Theorem 3.16 we obtain that the space $(\bar{X}, \bar{d}, \bar{\mu})$ is a PI-space with appropriate parameters. The constants in this theorem almost certainly could be substantially improved, but that is not relevant for us.

□

In the following appendix we give a result for tangents of RNP-Lipschitz differentiability spaces which can be seen as a corollary of the previous result. However, the alternative proof is simpler.

APPENDIX A. TANGENTS OF RNP-LIPSCHITZ DIFFERENTIABILITY SPACES

We give a shorter proof that tangents of RNP-Lipschitz differentiability spaces are almost everywhere PI-spaces. First a result on stability of the connectivity criterion.

Theorem 1.9. *If $(X_i, d_i, \mu_i, x_i) \rightarrow (X, d, \mu, x)$ is a convergent sequence of proper pointed metric measure spaces in the measured Gromov-Hausdorff sense, and each X_i is (C, δ, ϵ) -connected, then also the limit space (X, d, μ) is (C, δ', ϵ) -connected for every $\delta' > \delta$.*

Proof: It is sufficient to assume $\delta < \delta' < 1$, as otherwise the claim is trivial. Consider the spaces X_i, X embedded isometrically in some super-space (Z, d) , such that the induced measures converge weakly and X_i converge to X in the Hausdorff sense (see for similar argument [43]). Let $x, y \in X$ be arbitrary with $d(x, y) = r$ and $E \subset B(x, Cr)$ be a set such that

$$\mu(E \cap B(x, Cr)) < \epsilon \mu(B(x, Cr)).$$

There is a $0 < \epsilon' < \epsilon$, such that

$$\mu(E \cap B(x, Cr)) < \epsilon \mu(B(x, Cr)).$$

By regularity, no generality is lost by assuming that E is open. Define a function on X for each (fixed) $\eta > 0$ by $\rho_\eta(z) = \max\left(\frac{d(z, E^c)}{\eta}, 1\right)$.

Extend the function to Z in such a way that it is compactly supported, Lipschitz and bounded by 1, and denote the extension by the same symbol. By weak convergence

$$(A.1) \quad \int_Z \rho_\eta d\mu_i \rightarrow \int_Z \rho_\eta d\mu \leq \mu(E).$$

Thus, the open sets $E_i = \left\{ \rho_\eta > \frac{\epsilon'}{\epsilon} \right\} \cap X_i$ satisfy by (A.1)

$$(A.2) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \mu_n(E_n) &\leq \lim_{n \rightarrow \infty} \frac{\epsilon}{\epsilon'} \int_{X_n} \rho_\eta d\mu_i \\ &\leq \frac{\epsilon}{\epsilon'} \mu(E) < \epsilon \mu(B(x, Cr)). \end{aligned}$$

Further, choose sequences $x_i, y_i \in X_i$ such that $x_i \rightarrow x$ and $y_i \rightarrow y$. Our balls are assumed to be open, so from lower semi-continuity of the volume for open sets we see

$$\liminf_{n \rightarrow \infty} \mu(B(x_i, Cd(x_i, y_i))) \geq \mu(B(x, Cr)).$$

Finally, combining this with estimate (A.2) gives a N large enough such that for all $i > N$ it holds

$$\mu_i(E_i) < \epsilon \mu(B(x_i, Cd(x_i, y_i))).$$

We can now choose for all $i > N$ a 1-Lipschitz maps $\gamma_i: K_i \rightarrow X_i \subset Z$ defined on a compact subset $K_i \subset [0, 2Cd(x, y)]$, and satisfying

- $0 = \min(K_i)$ and $\max(K_i) \leq Cd(x_i, y_i) \leq 2Cd(x, y)$,
-
- $|[0, \max(K_i)] \setminus K_i| \leq \delta d(x_i, y_i)$, and
-

$$\rho_\eta(\gamma_i(t)) \leq \frac{\epsilon'}{\epsilon} < 1$$

for every $t \in K \setminus \{0, \max(K_i)\}$.

By an Arzela-Ascoli argument we can choose a sub-sequence i_k , a compact set K such that $d_H(K_{i_k}, K) \rightarrow 0$, and a map $\gamma: K \rightarrow X \subset Z$ which is a limit of γ_{i_k} . Also, for any $t_k \in K_{i_k}$ such that $t_k \rightarrow t$ we have $t \in K$ and $\gamma_{i_k}(t_k) \rightarrow \gamma(t)$. From this it is easy to see $\gamma(0) = x$. Further, as $\lim_{k \rightarrow \infty} \max(K_{i_k}) = \max(K)$, we have $\gamma(\max(K)) = y$. Finally, using the continuity of ρ_η on Z we get $\rho_\eta(\gamma(t)) \leq \frac{\epsilon'}{\epsilon} < 1$ for any $t \in K \setminus \{0, \max(K)\}$. This means that $\gamma(t) \notin \{\rho_\eta > \frac{\epsilon'}{\epsilon}\}$ for the same range of t .

By upper semi-continuity with respect to Hausdorff convergence of the volume for compact sets

$$\limsup_{k \rightarrow \infty} |K_{i_k}| \leq |K|,$$

and further

$$|[0, \max(K)] \setminus K| = \max(K) - |K| \leq \liminf_{k \rightarrow \infty} \max K_{i_k} - |K_{i_k}| \leq \delta d(x, y).$$

This shows $|\text{Undef}(\gamma)| \leq \delta d(x, y)$.

Let us now allow $\eta > 0$ to vary. For each η we can do the previous construction and define γ_η to satisfy the above conditions. Taking a sub-sequential limit with $\eta \rightarrow 0$, we get a 1-Lipschitz curve fragment γ with $|\text{Undef}(\gamma)| \leq \delta d(x, y)$ and $\text{len}(\gamma) \leq C(d, y)$. Further, for any $\eta > 0$ and any $t \in K_i \setminus \{\min(K_i), \max(K_i)\}$ we have $\rho_\eta(\gamma_\eta(t)) \leq \frac{\epsilon'}{\epsilon} < 1$. Thus, from the definition of ρ_η , we get

$$d(\gamma_\eta(t), E^c) \leq \eta \frac{\epsilon'}{\epsilon}.$$

Letting $\eta \rightarrow 0$ we get for the limiting curve $d(\gamma(t), E^c) \leq 0$ for $t \in K \setminus \{\min(K), \max(K)\}$, and thus $\gamma(t)$, when defined and excluding end-points, is not in E .

□

For the purposes of taking a tangent we need a slightly more general version of the previous theorem.

Theorem A.3. *If $(X_i, d_i, \mu_i, x_i) \rightarrow (X, d, \mu, p)$ is a convergent sequence of proper pointed metric measure and $S_i \subset X_i$ is a sequence of subsets such that X is $(C, \delta, \epsilon, r_i)$ -connected along S_i and S_i is ϵ_i -dense in $B(x_i, R_i)$ where $\lim_{i \rightarrow \infty} R_i = \infty$ and $\lim_{i \rightarrow \infty} \epsilon_i = 0$, $\lim_{i \rightarrow \infty} r_i = \infty$, then the limit space (X, d, μ) is (C, δ', ϵ) -connected for every $\delta' > \delta$.*

Proof: The proof proceeds exactly as before, and allows for constructing curve fragments γ_i in X that connect pairs of points in S_i , and taking their limits. The additional point we make, is that for any $x, y \in X$, there is a sequence of points $x_i, y_i \in S_i$ that converge to it. □

The following corollary would be a consequence of Theorem 1.1 but we provide an alternative and much simpler proof for it.

Corollary A.4. *Let (X, d, μ) be an RNP-differentiability space. Then X can be covered by positive measure subsets V_i , such that each V_i is metric doubling, when equipped with its restricted distance, and for μ -a.e. $x \in V_i$ each space $M \in T_x(V_i)$ admits a PI-inequality of type $(1, p)$ for all p sufficiently large.*

Proof: By [8] we have a decomposition into sets $K_i^j \subset V_i$ with $X = \bigcup V_i \cup N = \bigcup K_i^j \cup N'$, such that each V_i and $K_i^j \subset V_i$ satisfies the following,

- X is uniformly (D_i, r_i) -doubling along V_i .
- X is uniformly $(C_i, 2^{-30}, \epsilon_i, r_i)$ -connected along V_i .
- V_i is a uniform $(\frac{1}{2}, r_j^i)$ -density set in X along K_i^j .

Blowing up V_i with its restricted measure and distance, at a point of K_i^j , gives the desired result by Lemma A.3. □

Remark: The previous proof indicates the problem of applying the iterative procedure from Theorem 1.2 to conclude Poincaré inequalities on a RNP differentiability space. The constructed paths don't need to lie inside V_i , but instead only close by, and the closeness dictated by the density of the set and doubling. Since the curves may leave V_i , we can not guarantee the ability to refill their gaps.

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